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11 Textbook & Workbook

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Mind Action Series

M.D. Phillips, J. Basson & C. Botha

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MATHS GRADE 11 NCAPS TEXTBOOK INFORMATION

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The CONCEPT

The purpose of this publication is to cover all Maths topics in the new curriculum (CAPS) using a user-friendly, modern approach. The examples presented in each chapter of the textbook cover all of the main concepts in each topic. There is a logical, to-the-point, progression from one example to the next and the exercises reinforce the concepts inherent in the particular topic dealt with.

At the end of each chapter there is a mixed revision exercise. This exercise is for revising all of the concepts dealt with in the chapter. There is also a "Some Challenges" exercise which provides invaluable extension and problem solving for top learners. At the end of the book, short answers to all of the exercises have been provided.

ENDORSEMENT

"This textbook has always been a life-saver for me. I just don't have the time to waste trying to create my own lessons using other books and then still trying to get to assess my learners in the way that we are supposed to. This textbook has helped me to get through the content as quickly and effectively as possible leaving more time for me to assess my learners. I can also have a life outside of school and not get so wrapped up with so much work. I like the assessment tasks in the Teacher Guide. I have used them to meet the requirements in the policy documents."

Hester Jansen Van Vuuren, Educator

MATHEMATICS TEXTBOOK/WORKBOOK GRADE 11 NCAPS

CONTENTS

CHAPTER 1 – EXPONENTS AND SURDS

REVISION OF THE BASICS

The basic theory of exponents that you studied in Grade 10 is extremely important. We will now revise these important concepts.

EXPONENTIAL LAWS

Law	Example
1. $a^m . a^n = a^{m+n}$	$3^3 . 3^4 = 3^7$
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{2^8}{2^4} = 2^{8-4} = 2^4 = 16$
3. $(a^m)^n = a^{m \times n} = (a^n)^m$	$(3^3)^4 = 3^{3\times 4} = 3^{12} = (3^4)^3$
4. $(ab)^m = a^m b^m$	$(3x^4 y^3)^2 = 3^{1\times 2} . x^{4\times 2} . y^{3\times 2} = 9x^8 y^6$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{4}{x^2}\right)^3 = \frac{4^{1\times 3}}{x^{2\times 3}} = \frac{64}{x^6}$

EXPONENTIAL DEFINITIONS (COROLLARIES FROM THE LAWS)

Definition	Example
1. $a^0 = 1$	$3^0 = 1$
2. $1^a = 1$, with $a \in R$	$1^{2008} = 1$
3. $x^{-n} = \frac{1}{x^n}$	$x^{-4} = \frac{1}{x^4}$
4. $ax^{-n} = \frac{a}{x^n}$	$3x^{-4} = \frac{3}{x^4}$
5. $(ax)^{-n} = \frac{1}{(ax)^n}$	$(3x)^{-4} = \frac{1}{(3x)^4}$
6. $\frac{1}{x^{-n}} = x^n$	$\frac{1}{x^{-4}} = x^4$
7. $\frac{a}{x^{-n}} = ax^n$	$\frac{3}{x^{-4}} = 3x^4$
8. $\frac{1}{ax^{-n}} = \frac{x^n}{a}$	$\frac{1}{3x^{-4}} = \frac{x^4}{3}$
9. $\frac{1}{(ax)^{-n}} = (ax)^n$	$\frac{1}{(3x)^{-4}} = (3x)^4$
10. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$
11. $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$	$\frac{x^{-3}}{y^{-4}} = \frac{y^4}{x^3}$

EXAMPLE 1

Simplify the following:
(Try to identify the laws and/or the definitions that were used in each example)

(a) $2a^2.3a^3.b^0$

$= 6a^{2+3} \times 1$

$= 6a^5$

(b) $\dfrac{3^5.3.3^3}{3^2.3^6}$

$= \dfrac{3^5.3^1.3^3}{3^2.3^6}$

$= \dfrac{3^{5+1+3}}{3^{2+6}}$

$= \dfrac{3^9}{3^8} = 3^{9-8} = 3$

(c) $\dfrac{(-3a^3b)^2}{a^5b^3}$

$= \dfrac{9a^{3\times2}b^{1\times2}}{a^5b^3}$

$= \dfrac{9a^6b^2}{a^5b^3}$

$= 9a^{6-5}b^{2-3}$

$= 9ab^{-1} = \dfrac{9a}{b}$

(d) $\left(\dfrac{2x^3}{8y^{-4}}\right)^{-3}$

$= \left(\dfrac{8y^{-4}}{2x^3}\right)^3$

$= \left(\dfrac{4}{x^3y^4}\right)^3$

$= \dfrac{4^3}{x^{3\times3}y^{4\times3}} = \dfrac{64}{x^9y^{12}}$

EXAMPLE 2

Rewrite the following with prime bases and hence simplify without the use of a calculator:

(a) 16.8^{-4}

$= 2^4.(2^3)^{-4}$

$= 2^4.2^{-12}$

$= 2^{-8}$

$= \dfrac{1}{2^8}$

$= \dfrac{1}{256}$

(b) $\dfrac{9^4}{27}$

$= \dfrac{(3^2)^4}{3^3}$

$= \dfrac{3^8}{3^3}$

$= 3^5$

$= 243$

EXERCISE 1 **(REVISION OF GRADE 10 EXPONENTS)**

1. Simplify the following without using a calculator.

(a) $5a^0 + (5a)^0$ (b) $(-2x^3y^2)^3$ (c) $(3^3)^2.3^2$

(d) $\frac{2x^2y.4x^4y^3}{6x^6y^6}$ (e) $\left(\frac{-12x^3}{15y^{-2}}\right)^2$ (f) $\left(\frac{10y^4}{4y^6}\right)^{-2}$

(g) $\frac{(-3a^2b^4)^2}{3(a^2b)^3}$ (h) $(3x^3 \times 3x^3)^2 + (3x^3 + 3x^3)^2$

(i) $\frac{x+y}{x^{-1}+y^{-1}}$

2. Calculate the following without using a calculator:

(a) $\frac{1^{2011}}{2011^{-1}}$ (b) $(0{,}125)^{-3}$ (c) $\frac{6.2^{-2}}{3^{-3}}$

(d) $\left(\frac{1}{4}\right)^{-3x} \times 8^{2-2x}$ (e) $\frac{8.9^3}{32^2.27}$ (f) $2^{-4} + 2^4$

(g) $\frac{1}{3^{-1}+2^{-2}}$

MULTIPLICATION AND DIVISION OF POWERS WITH NUMERICAL BASES

EXAMPLE 3

Simplify: $\frac{4^{n+2}.9^{n-1}}{72^n.2^{1-n}}$

Solution

$\frac{4^{n+2}.9^{n-1}}{72^n.2^{1-n}}$ $72 = 2.36 = 2.6.6 = 2.(2.3).(2.3) = 2^3.3^2$

$= \frac{(2^2)^{n+2}.(3^2)^{n-1}}{(3^2.2^3)^n.2^{1-n}}$ Rewrite the powers with prime bases

$= \frac{2^{2n+4}.3^{2n-2}}{3^{2n}.2^{3n}.2^{1-n}}$ Multiply the exponents

$= \frac{2^{2n+4}.3^{2n-2}}{2^{3n+1-n}.3^{2n}}$ Add the exponents of like bases

$= \frac{2^{2n+4}.3^{2n-2}}{2^{2n+1}.3^{2n}}$

$= 2^{2n+4-(2n+1)}.3^{2n-2-(2n)}$ Subtract the exponents of like bases

$= 2^{2n+4-2n-1}.3^{-2}$

$= 2^3.3^{-2} = \frac{2^3}{3^2} = \frac{8}{9}$

ADDITION AND SUBTRACTION OF POWERS WITH NUMERICAL BASES

EXAMPLE 4

(a) Show that $2^{x+1}+3.2^x=5.2^x$

$2^{x+1}+3.2^x$

$=2^x.2^1+3.2^x$ Apply LAW 1: $a^{m+n}=a^m.a^n$

$=2^x(2+3)$ Factorise by taking out 2^x as a highest common factor

$=2^x(5)=5.2^x$

Please remember that when you are **factorising** power expressions, **the base and the exponent** have to be the same for it to be a **common factor.**

Alternatively:

$2^{x+1}+3.2^x=2^x.2^1+3.2^x$

If we let k represent the expression 2^x then

$2k+3k=5k$

$\therefore 2.2^x+3.2^x=5.2^x$

(b) Express $9^{x-1}+3^{2x}$ as a single power expression.

$9^{x-1}+3^{2x}$

$=(3^2)^{x-1}+3^{2x}$ Prime factorise the bases so as to work with equal bases

$=3^{2x-2}+3^{2x}$ Apply LAW 3: $(a^m)^n=a^{mn}$

$=3^{2x}.3^{-2}+3^{2x}$ Apply LAW 1: $a^{m+n}=a^m.a^n$

$=3^{2x}(3^{-2}+1)$

$=3^{2x}\left(\frac{1}{9}+1\right)=\left(\frac{10}{9}\right).3^{2x}$

(c) Simplify: $\dfrac{3^{x+2}-3^x}{3^{x+1}-3^{x-2}}$

$\dfrac{3^{x+2}-3^x}{3^{x+1}-3^{x-2}}$

$=\dfrac{3^x.3^2-3^x}{3^x.3^1-3^x.3^{-2}}$ Apply LAW 1: $a^{m+n}=a^m.a^n$

$=\dfrac{3^x(3^2-1)}{3^x(3^1-3^{-2})}$ Factorise numerator and denominator

$=\dfrac{9-1}{3-\frac{1}{3^2}}$ Cancel and simplify each bracket

$=\dfrac{8}{3-\frac{1}{9}}=\dfrac{36}{13}$

(d) Simplify $\frac{3^{2x}-1}{3^x+1}$

$\frac{3^{2x}-1}{3^x+1}$

$=\frac{(3^x)^2-1}{3^x+1}$ Apply LAW 3: $(a^m)^n=(a^n)^m$

$=\frac{(3^x+1)(3^x-1)}{(3^x+1)}=3^x-1$ Factorise the numerator

EXERCISE 2

1. Show that:

(a) $5.2^x-2^{x+2}=2^x$ (b) $9^x+3^{2x+1}=4.3^{2x}$

(c) $2^{2x-1}+4^{x+1}=2^{2x}\left(\frac{9}{2}\right)$ (d) $2^{x+3}-2^{x+2}=2^{x+2}$

(e) $2.10^x-5^{x+1}.2^x=-3.10^x$ (f) $4.3^{1-x}+3^{2-x}=\frac{21}{3^x}$

2. Simplify the following:

(a) $\frac{10^x.25^{x+1}}{5^x.50^{x-1}}$ (b) $\frac{6^{n+2}\times 10^{n-2}}{4^n\times 15^{n-2}}$ (c) $\frac{6^x.9^{x+1}.2}{27^{x+1}.2^{x-1}}$

(d) $\frac{8^n.6^{n-3}.9^{1-n}}{16^{n-1}.3^{-n}}$ (e) $\frac{2^{x+2}-2^{x+3}}{2^{x+1}-2^{x+2}}$ (f) $\frac{9^x+3^{2x+1}}{18^x.2^{1-x}}$

(g) $\frac{3.2^x-2^{x-1}}{2^x+2^{x+2}}$ (h) $\frac{4^x+2^{2x-1}}{2^{2x-1}}$ (i) $\frac{2.3^{x+2}+3^{x-3}}{5.3^{x-2}}$

3. Simplify the following:

(a) $\frac{2^{2x}-2^x}{2^x-1}$ (b) $\frac{2^{2x}-1}{2^x+1}$ (c) $\frac{9^x-9}{3^x-3}$

(d) $\frac{16-4^x}{2^x-4}$

POWERS WITH RATIONAL EXPONENTS

EXAMPLE 5

Simplify the following without using a calculator.

(a) $8^{\frac{2}{3}}$ The exponent is rational (a fraction)

$=(2^3)^{\frac{2}{3}}$

$=2^{3\times\frac{2}{3}}$ LAW 3

$=2^2$

$=4$

(b) $(0{,}25)^{\frac{3}{2}}$

Always change a decimal number to a fraction

$= \left(\frac{1}{4}\right)^{\frac{3}{2}}$ or alternatively $= \left(\frac{1}{4}\right)^{\frac{3}{2}}$

$= \left(\frac{1}{2^2}\right)^{\frac{3}{2}}$ $= \left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{1^{\frac{3}{2}}}{4^{\frac{3}{2}}}$

$= \left(2^{-2}\right)^{\frac{3}{2}} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ $= \frac{1}{\left(2^2\right)^{\frac{3}{2}}} = \frac{1}{2^3} = \frac{1}{8}$

EXERCISE 3

Simplify the following without using a calculator:

(a) $9^{\frac{1}{2}}$ (b) $16^{\frac{3}{4}}$ (c) $81^{\frac{1}{4}}$ (d) $27^{\frac{2}{3}}$

(e) $32^{\frac{3}{5}}$ (f) $64^{-\frac{1}{3}}$ (g) $\left(\frac{1}{27}\right)^{\frac{2}{3}}$ (h) $\left(\frac{8}{125}\right)^{-\frac{2}{3}}$

(i) $(0{,}125)^{\frac{1}{3}}$ (j) $(2{,}25)^{-\frac{1}{2}}$

ROOTS AND SURDS

It is easy to understand that 8^2 means $8 \times 8 = 64$. The exponent in this case is an integer. If the exponent of a power is a rational number (fraction), the meaning is slightly different.

Consider, for example, the number $8^{\frac{2}{3}}$. If we determine the value of $8^{\frac{2}{3}}$ and $\sqrt[3]{8^2}$, we can understand the meaning of rational exponents.

$8^{\frac{2}{3}}$ and $\sqrt[3]{8^2}$

$= (2^3)^{\frac{2}{3}}$ $= \sqrt[3]{64}$

$= 2^{3 \times \frac{2}{3}}$ $= 4$

$= 2^2$

$= 4$

Therefore, it is clear that: $8^{\frac{2}{3}} = \sqrt[3]{8^2} =$ the cube root of 8^2

In general, we have the following definition: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$, with $\frac{m}{n}$ in its simplest form. (Note that the denominator n of the rational exponent refers to the n^{th} root)

The Root Definition: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

It is extremely important that you are able to work with the Root Definition in both directions.

EXAMPLE 6

Write the following without the root sign.

(a) $\sqrt[3]{a^2} = a^{\frac{2}{3}}$ (b) $\sqrt{x} = \sqrt[2]{x^1} = x^{\frac{1}{2}}$

EXAMPLE 7

Write the following with a root sign.

(a) $a^{\frac{2}{5}} = \sqrt[5]{a^2}$ (b) $x^{\frac{4}{3}} = \sqrt[3]{x^4}$

EXAMPLE 8

Simplify the following without using a calculator.

(a) $\sqrt[4]{\left(\frac{625}{81}\right)^3}$

$= \left(\frac{625}{81}\right)^{\frac{3}{4}}$

$= \left(\frac{5^4}{3^4}\right)^{\frac{3}{4}}$

$= \frac{5^{4\times\frac{3}{4}}}{3^{4\times\frac{3}{4}}}$

$= \frac{5^3}{3^3} = \frac{125}{27}$

(b) $\sqrt[3]{\left(3\frac{3}{8}x^{12}y^{-6}\right)^2}$

$= \left(\frac{27}{8}x^{12}y^{-6}\right)^{\frac{2}{3}}$

$= \left(\frac{3^3x^{12}}{2^3y^6}\right)^{\frac{2}{3}}$

$= \frac{3^{3\times\frac{2}{3}}x^{12\times\frac{2}{3}}}{2^{3\times\frac{2}{3}}y^{6\times\frac{2}{3}}} = \frac{3^2x^8}{2^2y^4} = \frac{9x^8}{4y^4}$

EXERCISE 4

1. Write the following without the root sign.

   (a) $\sqrt[4]{a^6}$ (b) $\sqrt[3]{x^5}$ (c) $\sqrt{a^5}$ (d) $\sqrt[4]{x^3}$

   (e) $\sqrt[3]{a^2}$ (f) $\sqrt{x}$

2. Write the following with a root sign.

   (a) $a^{\frac{4}{5}}$ (b) $x^{\frac{1}{6}}$ (c) $a^{\frac{5}{2}}$ (d) $x^{\frac{2}{3}}$

   (e) $a^{\frac{1}{2}}$ (f) $x^{\frac{3}{2}}$

3. Simplify the following without using a calculator.

   (a) $\sqrt[3]{8^4}$ (b) $\sqrt[10]{32^2}$ (c) $\sqrt[3]{27^2}$

   (d) $\sqrt{\left(\frac{1}{4}\right)^3}$ (e) $\sqrt[3]{(0{,}125)^{-2}}$ (f) $\left(\sqrt{2}\right)^{\frac{4}{3}}$

   (g) $\left(\sqrt{3}\right)^3.3^{\frac{1}{2}}$ (h) $\sqrt[4]{a^2}.\sqrt{a}$ (i) $x^{\frac{1}{4}}.(x^{\frac{3}{8}})^2$

   (j) $(\sqrt{8x^6})^4$

4. Simplify as far as possible and leave your answers in positive exponential form.

(a) $\sqrt{(0{,}04)^5}$ (b) $\dfrac{8^{x+1}.2^{1-x}}{(\sqrt{2})^{4x-2}}$ (c) $\sqrt[3]{27x^6}+\sqrt{81x^4}-\sqrt[4]{256x^8}$

(d) $\left(\dfrac{4}{9}\right)^{-\frac{3}{2}}$ (e) $\dfrac{\left(81k^{-2}\right)^{\frac{1}{4}}.k^{\frac{1}{2}}}{\left(27k\right)^{-1}}$ (f) $\dfrac{(\sqrt{8}.2^{-3})^4}{64^{-\frac{1}{2}}}$

5. Without using a calculator, simplify $4^{\frac{3}{2}}+8^{\frac{1}{3}}-\left(0{,}5\right)^{-2}$

SURDS

When a root is irrational it is referred to as a surd. In other words a surd is the root of a number that cannot be determined exactly. Thus $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{20}$ are surds but $\sqrt{4}$, $\sqrt{36}$ and $\sqrt[3]{27}$ are not.

There are two surd laws for $a>0$, $b>0$, $n\geq 2$, $n\in$natural numbers.

Law 1 (multiplication law): $\sqrt[n]{a}.\sqrt[n]{b}=\sqrt[n]{a.b}$ or $\sqrt[n]{a.b}=\sqrt[n]{a}.\sqrt[n]{b}$

Law 2 (division law) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$ or $\sqrt[n]{\dfrac{a}{b}}=\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

EXAMPLE 9 **(Multiplication and division of surds)**

Simplify the following:

(a) $\sqrt{2}.\sqrt{3}=\sqrt{2.3}=\sqrt{6}$ (b) $\sqrt{3}.\sqrt{3}=\sqrt{9}=3$

(c) $(\sqrt{4})^2=\sqrt{4}.\sqrt{4}=\sqrt{16}=4$ (d) $\dfrac{\sqrt{27}}{\sqrt{3}}=\sqrt{\dfrac{27}{3}}=\sqrt{9}=3$

(e) $\dfrac{\sqrt[3]{16}}{\sqrt[3]{2}}=\sqrt[3]{\dfrac{16}{2}}=\sqrt[3]{8}=2$

From the above examples it is **important to notice** that:

$(\sqrt{a})^2=\sqrt{a}.\sqrt{a}=a$ (see examples (b) and (c) above) and therefore:

(f) $(5\sqrt{3})^2=5^2.(\sqrt{3})^2=25.3=75$

EXAMPLE 10 **(Addition and subtraction of surds)**

Simplify the following:

(a) $5\sqrt{3}+2\sqrt{3}-3\sqrt{3}$ Think of this as $5x+2x-3x=4x$

$=4\sqrt{3}$

(b) $\sqrt{7}+2\sqrt{7}-6\sqrt{7}$

$=1\sqrt{7}+2\sqrt{7}-6\sqrt{7}$

$=-3\sqrt{7}$

(c) $\sqrt[3]{5}+5\sqrt[3]{5}-2\sqrt[3]{5}$

$=4\sqrt[3]{5}$

EXAMPLE 11

(a) Write in simplest surd form and simplify without using a calculator:

(1) $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$ (2) $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4}.\sqrt{2} = 2\sqrt{2}$

(3) $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2} = 3\sqrt[3]{2}$

(b) Simplify the following and leave your answer in **simplest surd** form.

(1) $\sqrt{12} + 4\sqrt{75}$

$= \sqrt{4 \times 3} + 4.\sqrt{25 \times 3}$

$= \sqrt{4}.\sqrt{3} + 4.\sqrt{25}.\sqrt{3}$

$= 2\sqrt{3} + 4.5\sqrt{3}$

$= 2\sqrt{3} + 20\sqrt{3}$

$= 22\sqrt{3}$

(2) $\left(2 + 3\sqrt{2}\right)^2$

$= \left(2 + 3\sqrt{2}\right)\left(2 + 3\sqrt{2}\right)$

$= 4 + 2(3\sqrt{2}) + 2(3\sqrt{2}) + \left(3\sqrt{2}\right).\left(3\sqrt{2}\right)$

$= 4 + 6\sqrt{2} + 6\sqrt{2} + 9.2$

$= 4 + 12\sqrt{2} + 18$

$= 22 + 12\sqrt{2}$

(3) $\dfrac{\sqrt{48} - \sqrt{3}}{\sqrt{27}}$

$= \dfrac{\sqrt{16.3} - \sqrt{3}}{\sqrt{9.3}}$

$= \dfrac{4\sqrt{3} - \sqrt{3}}{3\sqrt{3}}$

$= \dfrac{3\sqrt{3}}{3\sqrt{3}} = 1$

(4) $\dfrac{\sqrt{4x^9} - \sqrt{16x^9}}{\sqrt{x}}$

$= \dfrac{2\sqrt{x^8.x} - 4\sqrt{x^8.x}}{\sqrt{x}}$

$= \dfrac{2x^4\sqrt{x} - 4x^4\sqrt{x}}{\sqrt{x}}$

$= \dfrac{-2x^4\sqrt{x}}{\sqrt{x}}$

$= -2x^4$

or alternatively

$= \dfrac{2x^{\frac{9}{2}} - 4x^{\frac{9}{2}}}{x^{\frac{1}{2}}}$

$= \dfrac{-2x^{\frac{9}{2}}}{x^{\frac{1}{2}}}$

$= -2x^{\frac{9}{2}-\frac{1}{2}}$

$= -2x^{\frac{8}{2}}$

$= -2x^4$

EXERCISE 5

1. Simplify the following without using a calculator:

(a) $\sqrt{7}\sqrt{3}$ (b) $\sqrt{7}\sqrt{7} + \left(\sqrt{11}\right)^2$ (c) $\left(2\sqrt{3}\right)^2$

(d) $3\sqrt{6} - \sqrt{6} + 7\sqrt{6}$ (e) $\sqrt[4]{3} + 7\sqrt[4]{3} - 5\sqrt[4]{3}$ (f) $\sqrt{3} + \sqrt{27}$

(g) $2\sqrt{18} - \sqrt{32}$ (h) $\dfrac{\sqrt{32}}{\sqrt{2}}$ (i) $\dfrac{\sqrt[3]{51}}{\sqrt[3]{3}}$

(j) $\dfrac{\sqrt{32}}{2}$ (k) $\dfrac{\sqrt{54}}{9}$ (l) $\dfrac{\sqrt{8} + \sqrt{2}}{\sqrt{2}}$

(m) $\dfrac{\sqrt{50} - \sqrt{2}}{\sqrt{8}}$ (n) $\dfrac{\sqrt[3]{16} - \sqrt[3]{54}}{\sqrt[3]{2}}$ (o) $\dfrac{\sqrt{48} - \sqrt{32}}{\sqrt{12} - \sqrt{8}}$

(p) $\left(5 - \sqrt{3}\right)\left(5 + \sqrt{3}\right)$ (q) $\left(5 - \sqrt{3}\right)^2$ (r) $\left(5\sqrt{2} - \sqrt{3}\right)^2$

(s) $2\sqrt{5}\left(\sqrt{5} - 3\sqrt{20}\right)$ (t) $\left(\sqrt{5} + 2\sqrt{3}\right)^2 - \left(\sqrt{5} - 2\sqrt{3}\right)^2$

(u) $2\sqrt{8m}\left(\sqrt{2m} - 3\sqrt{18m}\right)$ (v) $\dfrac{10\sqrt{2x^{20}} + 7\sqrt{8x^{20}}}{\sqrt{18x^{20}}}$

2. Write the following in simplest surd form without the use of a calculator:
   (a) $\sqrt{2^{11}}$ (b) $\sqrt[3]{2^{11}}$ (c) $\sqrt{405x^7}$

3. The length of a rectangle is $\sqrt{3}+1$ and its breadth is $\sqrt{3}-1$. Determine the length of its diagonal in surd form.

4. Show that $\dfrac{\sqrt{12}-\sqrt{75}}{3^{\frac{3}{2}}} = -1$ without using a calculator.

5. Without using a calculator, show that $\dfrac{\left(\sqrt{2}-1\right)^2}{6-\sqrt{32}} = \dfrac{1}{2}$

6. Simplify $\dfrac{\sqrt{75}-\sqrt{50}}{\left(\sqrt{10}-\sqrt{5}\right)\left(\sqrt{10}+\sqrt{5}\right)}$

RATIONALISING THE DENOMINATOR

The principle of rationalising the denominator is to remove the surds (irrational numbers) in the denominator.

EXAMPLE 12

Rationalise the denominator of:

(a) $\dfrac{6}{\sqrt{3}}$ (b) $\dfrac{8}{\sqrt[3]{2}}$ (c) $\dfrac{5}{4-\sqrt{3}}$

Solutions

(a) $\dfrac{6}{\sqrt{3}}$

$= \dfrac{6}{\sqrt{3}} \times 1$

$= \dfrac{6}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$

(b) $\dfrac{8}{\sqrt[3]{2}} = \dfrac{8}{\sqrt[3]{2}} \times 1 \times 1$

$= \dfrac{8}{\sqrt[3]{2}} \times \dfrac{\sqrt[3]{2}}{\sqrt[3]{2}} \times \dfrac{\sqrt[3]{2}}{\sqrt[3]{2}}$

$= \dfrac{8\sqrt[3]{4}}{\sqrt[3]{8}}$

$= \dfrac{8.\sqrt[3]{4}}{2}$

$= 4\sqrt[3]{4}$

(c) $\dfrac{5}{4-\sqrt{3}}$

$= \dfrac{5}{\left(4-\sqrt{3}\right)} \times \dfrac{\left(4+\sqrt{3}\right)}{\left(4+\sqrt{3}\right)}$

$= \dfrac{5\left(4+\sqrt{3}\right)}{4^2-\left(\sqrt{3}\right)^2}$

$= \dfrac{5\left(4+\sqrt{3}\right)}{16-3}$

$= \dfrac{20+5\sqrt{3}}{13}$

EXERCISE 6

1. Rationalise the denominator of the following and leave the answer in simplest surd form:

   (a) $\dfrac{3}{\sqrt{6}}$ (b) $\dfrac{\sqrt{8}}{\sqrt{3}}$ (c) $\dfrac{2}{5\sqrt{2}}$

   (d) $\dfrac{3\sqrt{2}}{\sqrt{8}}$ (e) $\dfrac{3}{\sqrt[3]{9}}$

2. (a) $\dfrac{2}{1-\sqrt{3}}$ (b) $\dfrac{4}{\sqrt{7}-2}$ (c) $\dfrac{\sqrt{2}}{4+\sqrt{2}}$

3. Show that $\dfrac{\sqrt{x}}{x}+\dfrac{y}{\sqrt{x}}$ can be written as $\dfrac{\sqrt{x}(y+1)}{x}$

EXPONENTIAL EQUATIONS

In an exponential equation, **the exponent is the unknown.** For example, consider the equation $3^x = 9$. The equation can be solved as follows:

$3^x = 9$

$\therefore 3^x = 3^2$ Write 9 to the base 3

$\therefore x = 2$ Equate the exponents

Also take note that $a^x > 0$ for all real values of x where $a > 0$ and $a \neq 1$

Consider, for example, the expression 3^x.

For $x = -2$ $3^x = 3^{-2} = \dfrac{1}{9} > 0$ For $x = -1$ $3^x = 3^{-1} = \dfrac{1}{3} > 0$

For $x = 0$ $3^x = 3^0 = 1 > 0$ For $x = 1$ $3^x = 3^1 = 3 > 0$

This means that for the expression 3^x there is no real value of x for which $3^x < 0$

The equation $3^x = -3$ has no solution since $3^x > 0$ for all real values of x.

EXAMPLE 13 (Revision of Grade 10 work)

Solve for x:

(a) $2^x = \dfrac{1}{16}$

$\therefore 2^x = \dfrac{1}{2^4}$

$\therefore 2^x = 2^{-4}$

$\therefore x = -4$

(b) $3.5^x = 0,6$

$\therefore 3.5^x = \dfrac{6}{10}$ Write decimal as a fraction

$\therefore 3.5^x = \dfrac{3}{5}$

$\therefore 5^x = \dfrac{1}{5}$

$\therefore 5^x = 5^{-1}$

$\therefore x = -1$ Equate the exponents

(c) $9^{x+1} = 27^x$

$\therefore (3^2)^{x+1} = (3^3)^x$ Same base

$\therefore 3^{2x+2} = 3^{3x}$

$\therefore 2x + 2 = 3x$

$\therefore -x = -2$

$\therefore x = 2$

(d) $16^x.2^{x+1} = \sqrt[5]{4}$

$\therefore (2^4)^x.2^{x+1} = 4^{\frac{1}{5}}$ Apply $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$\therefore 2^{4x}.2^{x+1} = (2^2)^{\frac{1}{5}}$ Same base

$\therefore 2^{5x+1} = 2^{\frac{2}{5}}$

$\therefore 5x + 1 = \frac{2}{5}$

$\therefore 5x = -\frac{3}{5}$

$\therefore x = \frac{-3}{25}$

EXERCISE 7

Solve for x:

(a) $3^x = 1$ (b) $2^x = \sqrt{2}$ (c) $4^x = 8$

(d) $\left(\frac{1}{9}\right)^x = 27$ (e) $16.16^x = 64$ (f) $3^{x+1} = \frac{\sqrt{3}}{3}$

(g) $4\left(\frac{1}{4}\right)^{x-1} = 8$ (h) $\frac{1}{3}(3)^{x-1} = \frac{1}{27}$ (i) $8^x.16^{x-1} = 1$

(j) $\frac{3^{2x-1}}{3^x} = 3$ (k) $\sqrt[3]{9} = \left(\sqrt{3}\right)^{2x}$ (l) $(0{,}375)^x = \frac{9}{64}$

(m) $8^{-x} - 2.4^{x-1} = 0$ (n) $6^x + 6^x + 6^x + 6^x + 6^x + 6^x = 6^{6x}$

EXAMPLE 14 **(Examples which involve factorisation)**

(a) (1) Show that $5.2^x + 3.2^{x+2} = 17.2^x$

(2) Hence solve $5.2^x + 3.2^{x+2} = 68$

(b) Solve for x:

(1) $9^x + 3^{2x+1} = 4\sqrt{3}$

(2) $3^{2x} + 6.3^x - 27 = 0$

Solutions

(a) (1) $5.2^x + 3.2^{x+2}$

$= 5.2^x + 3.2^x.2^2$ Apply LAW 1: $a^{m+n} = a^m.a^n$

$= 2^x\left(5 + 3.2^2\right)$

$= 2^x\left(5 + 12\right) = 17.2^x$

(2) $5.2^x + 3.2^{x+2} = 68$

$\therefore 17.2^x = 68$ since $5.2^x + 3.2^{x+2} = 17.2^x$

$\therefore 2^x = 4$

$\therefore 2^x = 2^2$

$\therefore x = 2$

(b) (1) $9^x + 3^{2x+1} = 4\sqrt{3}$

$\therefore (3^2)^x + 3^{2x+1} = 4.3^{\frac{1}{2}}$ Express as powers with prime bases

$\therefore 3^{2x} + 3^{2x}.3^1 = 4.3^{\frac{1}{2}}$ Apply LAW 1: $a^{m+n} = a^m . a^n$

$\therefore 3^{2x}(1+3) = 4.3^{\frac{1}{2}}$ Factorise

$\therefore 4.3^{2x} = 4.3^{\frac{1}{2}}$

$\therefore 3^{2x} = 3^{\frac{1}{2}}$

$\therefore 2x = \frac{1}{2}$

$\therefore x = \frac{1}{4}$

(2) $3^{2x} + 6.3^x - 27 = 0$

Note that the coefficients of the terms in x in the exponents are different. This is a "disguised" trinomial.

$\therefore (3^x)^2 + 6.3^x - 27 = 0$

$\therefore (3^x - 3)(3^x + 9) = 0$

$\therefore 3^x - 3 = 0$ or $3^x + 9 = 0$

$\therefore 3^x = 3$ or $3^x = -9$

$\therefore x = 1$ no solution since $3^x > 0$ for all real values of x

> Note: It might be helpful to revise the factorisation of quadratic expressions before discussing this example (see Chapter 2 page 26)

Alternatively, you can use a method referred to as the 'k'-substitution method.

$\therefore 3^{2x} + 6.3^x - 27 = 0$

Let $3^x = k$

$\therefore 3^{2x} = k^2$

$\therefore k^2 + 6k - 27 = 0$

$\therefore (k-3)(k+9) = 0$

$\therefore k = 3$ or $k + 9 = 0$

or $k = -9$

$\therefore 3^x = 3$ or $3^x = -9$ substitute $k = 3^x$

$\therefore x = 1$ no solution ($3^x > 0$ for all values of x)

EXERCISE 8

1. Solve for x:

(a) $4.3^x + 3^x = 15$ (b) $2^{x+1} + 2^{x-1} = 5$

(c) $2^{x+2} + 2^{x-2} + 2^x = 84$ (d) $3\left(3^{x+2} + 3^{x-1}\right) = 84$

(e) $3^{2x} - 2.3^{x+2} + 81 = 0$ (f) $2^{2x} - 6.2^{x-1} + 2 = 0$

(g) $4^x + 4.2^x - 5 = 0$ (h) $9^x - 3^{x-2} = 0$

(k) $3^{2x+1} - 4.3^{x+2} + 81 = 0$ (l) $16^x + 2^{2x+1} - 2^3 = 0$

(m) $\left(\frac{1}{4}\right)^x - 3.2^{-x} - 4 = 0$ (n) $2^x + 16.2^{-x} - 10 = 0$

2. (a) Show that $4.3^{1-x} + 3^{2-x} = \frac{21}{3^x}$

   (b) Hence or otherwise solve for x if $4.3^{1-x} + 3^{2-x} = 63$

3. (a) Show that $\sqrt{2^x} + \sqrt{2^{x+4}} = 5.(\sqrt{2})^x$

   (b) Hence or otherwise solve for x if $\sqrt{2^x} + \sqrt{2^{x+4}} = 2\frac{1}{2}$

EQUATIONS WITH RATIONAL EXPONENTS

EXAMPLE 15

Consider the equation $x^{\frac{1}{2}} = 4$.

By raising both sides to the power 2 we will be able to solve for x.

$x^{\frac{1}{2}} = 4$

$\therefore (x^{\frac{1}{2}})^2 = 4^2$

$\therefore x = 16$

Consider the equation $x^{\frac{1}{3}} = 3$.

By raising both sides to the power 3 we will be able to solve for x.

$x^{\frac{1}{3}} = 3$

$\therefore (x^{\frac{1}{3}})^3 = 3^3$

$\therefore x = 27$

Consider the equation $x^{\frac{1}{2}} = -4$

$\therefore (x^{\frac{1}{2}})^2 = (-4)^2$

$\therefore x = 16$

However, if you check the solution in the original equation, you will notice a problem.

$\text{LHS} = 16^{\frac{1}{2}} = (16)^{\frac{1}{2}} = (2^4)^{\frac{1}{2}} = 2^2 = 4$

$\text{RHS} = -4$

$\therefore \text{LHS} \neq \text{RHS}$

It is therefore clear that $x^{\frac{1}{2}} \neq -4$

Note: $x^{\frac{1}{\text{even number}}} \neq$ negative number

EXAMPLE 16

Solve for x:

(a) $x^{\frac{1}{4}} = 3$ (b) $x^{\frac{1}{4}} = -3$ (c) $x^{\frac{1}{3}} = -3$

Solutions

(a) $x^{\frac{1}{4}} = 3$

$\therefore (x^{\frac{1}{4}})^4 = 3^4$

$\therefore x = 81$

(b) $x^{\frac{1}{4}} = -3$

no solution (even denominator)

(c) $x^{\frac{1}{3}} = -3$

Here we don't have an even number in the denominator of the fractional exponent. Proceed as normal.

$\therefore (x^{\frac{1}{3}})^3 = (-3)^3$

$\therefore x = -27$

By checking this answer you will notice that this answer does in fact satisfy the equation:

$x^{\frac{1}{3}} = \sqrt[3]{x} = -3$

If $x = -27$ then $\sqrt[3]{x} = \sqrt[3]{-27} = -3$

> **Note:** Whenever you have equations of the form $x^{\frac{m}{n}} = \text{number}$, always rewrite the expression $x^{\frac{m}{n}}$ in the form $(x^{\frac{1}{n}})^m$.

EXAMPLE 17

Solve for x:

(a) $x^{\frac{2}{3}} = 4$ (b) $x^{\frac{3}{2}} = 8$ (c) $x^{\frac{2}{3}} = -4$

(d) $x^{\frac{3}{2}} = -8$ (e) $3x^{-\frac{2}{3}} = 27$ (f) $x^{\frac{1}{2}} + x^{\frac{1}{4}} - 2 = 0$

Solutions

(a) $x^{\frac{2}{3}} = 4$

$\therefore (x^{\frac{1}{3}})^2 = 4$

Let $k = x^{\frac{1}{3}}$

$\therefore k^2 = 4$

$\therefore k^2 - 4 = 0$

$\therefore (k-2)(k+2) = 0$

$\therefore k = 2$ or $k = -2$

$\therefore x^{\frac{1}{3}} = 2$ or $x^{\frac{1}{3}} = -2$

$\therefore (x^{\frac{1}{3}})^3 = (2)^3$ or $(x^{\frac{1}{3}})^3 = (-2)^3$

$\therefore x = 8$ or $x = -8$

(b) $x^{\frac{3}{2}} = 8$

$\therefore (x^{\frac{1}{2}})^3 = 8$

Let $k = x^{\frac{1}{2}}$

$\therefore k^3 = 8$

$\therefore k = 2$

$\therefore x^{\frac{1}{2}} = 2$

$\therefore (x^{\frac{1}{2}})^2 = 2^2$

$\therefore x = 4$

(c) $x^{\frac{2}{3}} = -4$

$\therefore (x^{\frac{1}{3}})^2 = -4$

Let $k = x^{\frac{1}{3}}$

$\therefore k^2 = -4$

$\therefore$ no real solution

(d) $x^{\frac{3}{2}} = -8$

$\therefore (x^{\frac{1}{2}})^3 = -8$

Let $k = x^{\frac{1}{2}}$

$\therefore k^3 = -8$

$\therefore k = -2$

$\therefore x^{\frac{1}{2}} = -2$

$\therefore$ no real solution

(e) $3x^{-\frac{2}{3}} = 27$

$\therefore x^{-\frac{2}{3}} = 9$

$\therefore (x^{-\frac{1}{3}})^2 = 9$

Let $k = x^{-\frac{1}{3}}$

$\therefore k^2 = 9$

$\therefore k^2 - 9 = 0$

$\therefore (k+3)(k-3) = 0$

$\therefore k = -3 \quad \text{or} \quad k = 3$

$\therefore x^{-\frac{1}{3}} = -3 \quad \text{or} \quad x^{-\frac{1}{3}} = 3$

$\therefore (x^{-\frac{1}{3}})^{-3} = (-3)^{-3} \quad \text{or} \quad (x^{-\frac{1}{3}})^{-3} = (3)^{-3}$

$\therefore x = \dfrac{1}{(-3)^3} \quad \text{or} \quad x = \dfrac{1}{(3)^3}$

$\therefore x = -\dfrac{1}{27} \quad \text{or} \quad x = \dfrac{1}{27}$

(f) $x^{\frac{1}{2}} + x^{\frac{1}{4}} - 2 = 0$

Let $k = x^{\frac{1}{4}}$

$\therefore k^2 = (x^{\frac{1}{4}})^2 = x^{\frac{1}{2}}$

$\therefore k^2 + k - 2 = 0$

$\therefore (k+2)(k-1) = 0$

$\therefore k = -2 \quad \text{or} \quad k = 1$

$\therefore x^{\frac{1}{4}} = -2 \quad \text{or} \quad x^{\frac{1}{4}} = 1$

$\therefore$ no solution $\qquad x = 1$

EXERCISE 9

1. Solve for x:

(a)	$x^{\frac{1}{2}} = 3$	(b)	$x^{\frac{1}{2}} = -3$	(c)	$x^{\frac{1}{3}} = 3$	(d)	$x^{\frac{1}{3}} = -3$
(e)	$x^{\frac{1}{4}} = 2$	(f)	$x^{\frac{1}{4}} = -2$	(g)	$x^{\frac{2}{3}} = 64$	(h)	$x^{\frac{2}{3}} = -64$
(i)	$x^{\frac{3}{2}} = 125$	(j)	$x^{\frac{3}{2}} = -125$	(k)	$x^{\frac{3}{5}} = 8$	(l)	$x^{\frac{3}{5}} = -8$
(m)	$x^{-\frac{1}{2}} = 1\frac{1}{3}$	(n)	$x^{-\frac{2}{3}} = 16$	(o)	$x^{-\frac{3}{2}} = 27$	(p)	$2.\sqrt[3]{x^2} = 8$
(q)	$3x^3 - 81 = 0$	(r)	$\sqrt[3]{\dfrac{1}{x}} = 9$				

2. Solve for x:

(a)	$x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + 2 = 0$	(b)	$x - 5\sqrt{x} + 4 = 0$	(c)	$x^{\frac{1}{2}} - 2x^{\frac{1}{4}} - 3 = 0$
(d)	$4x^{\frac{2}{3}} + 5x^{\frac{1}{3}} - 6 = 0$	(e)	$x^{-1} - 7x^{-\frac{1}{2}} = 18$	(f)	$x^{\frac{4}{3}} - \sqrt[3]{x^2} - 72 = 0$

SIMPLE SURD EQUATIONS

Before dealing with simple surd equations, it is important to discuss the concept of non-real numbers. In Grade 10, you studied real numbers (rational and irrational numbers). We will now introduce non-real numbers.

If $x \geq 0$, then the expression $\sqrt{x}$ will be a real number.

For example, $\sqrt{4}\,;\sqrt{5}\,;-\sqrt{6}\,;\sqrt{0}$ are real numbers since the numbers under the square root signs are positive.

Please note that $\sqrt{9}=3$ and NOT ± 3 (the sign $\sqrt{\ }$ is defined to be the positive square root of 9).

However, if $x<0$, then the expression $\sqrt{x}$ will be a **non-real** number.

For example, $\sqrt{-1}\,;\sqrt{-4}\,;\sqrt{-5}$ are non-real numbers since the numbers under the square root signs are negative.

Some important principles are worth mentioning:

Principle 1

The expression $\sqrt{A}$ will only be **real** if $A \geq 0$.
If $A<0$, then the expression will be non-real.

Principle 2

$\sqrt{A} \neq$ negative number

Principle 3

$\left(\sqrt{A}\right)^2 = A$

EXAMPLE 18

Solve for x, if possible:

(a) $\sqrt{x-1}-3=0$ (b) $\sqrt{x-1}+3=0$ (c) $\sqrt{2-7x}+2x=0$

Solutions

(a)
$$\sqrt{x-1}-3=0$$
$$\therefore \sqrt{x-1}=3$$
$$\therefore \left(\sqrt{x-1}\right)^2=3^2$$
$$\therefore x-1=9$$
$$\therefore x=10$$

(b)
$$\sqrt{x-1}+3=0$$
$$\therefore \sqrt{x-1}=-3$$
$\therefore$ no solution

(c)
$$\sqrt{2-7x}+2x=0$$
$$\therefore \sqrt{2-7x}=-2x$$
$$\therefore \left(\sqrt{2-7x}\right)^2=(-2x)^2$$
$$\therefore 2-7x=4x^2$$
$$\therefore 0=4x^2+7x-2$$
$$\therefore 0=(4x-1)(x+2)$$
$$\therefore x=\frac{1}{4} \quad \text{or} \quad x=-2$$

It is important to check whether each answer is actually a solution to the equation.

For $x = \frac{1}{4}$

LHS

$= \sqrt{2-7x}$

$= \sqrt{2-7\left(\frac{1}{4}\right)} = \sqrt{2-\frac{7}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

RHS $= -2x = -2\left(\frac{1}{4}\right) = -\frac{1}{2}$

$\therefore$ LHS $\neq$ RHS

$\therefore x = \frac{1}{4}$ is not a solution

For $x = -2$

LHS $= \sqrt{2-7(-2)} = \sqrt{16} = 4$

RHS $= -2(-2) = 4$

$\therefore$ LHS = RHS

$\therefore x = -2$ is the solution

<u>EXERCISE 10</u>

Solve for x, if possible:

(a) $\sqrt{x-4} - 5 = 0$ (b) $\sqrt{x-4} + 5 = 0$ (c) $\sqrt{5x+6} = x$

(d) $\sqrt{3x-2} - x = 0$ (e) $x + \sqrt{-4x-3} = 0$ (f) $\sqrt{x+2} + 4 = x$

(g) $\sqrt{4-2x} - \sqrt{x+1} = 0$ (h) $\sqrt{5-x} + 1 + x = 0$ (i) $\sqrt{x+5}\,.\sqrt{x-2} = 3\sqrt{2}$

<u>REVISION EXERCISE</u>

1. Simplify the following without the use of a calculator:

(a) $\left(\frac{1}{2}\right)^{x-1}.(\sqrt{2})^{2x-1}$ (b) $\frac{2.3^{x} - 3^{x-2}}{2.3^{x-2}}$

(c) $\frac{6^{x} \times 3^{x-2}}{6.18^{x-2}}$ (d) $\left(\sqrt{2}-3\right)\left(\sqrt{8}+2\right)$

(e) $\frac{\sqrt{20x^5} \times \sqrt{10x} + \sqrt{2x^6}}{\sqrt{2x^6}}$ (f) $\frac{\sqrt[4]{16x^5}}{\sqrt{\sqrt{x}}}$ (g) $\sqrt{72} - \sqrt{2^7}$

2. Simplify the following: $\frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{2}$ (rationalise the denominator)

3. Solve for x if: $3^{3x} + 3^{3x} + 3^{3x} = 3^{9x}$

4. Solve the following equations:

(a) $x^{\frac{1}{3}} = 9$ (b) $\left(\frac{1}{3}\right)^{x} = 9$ (c) $\frac{1}{3}x = 9$

(d) $\frac{1}{16}x^{\frac{3}{4}} = 4$ (e) $\frac{1}{16}.2^{\frac{3}{4}x} = 4$ (f) $4x^{\frac{2}{3}} + 2 = 18$

5. Solve for x:

(a) $2^{x+1} + 2^{x+2} = 6$ (b) $2^{x+1}.2^{x+2} = 4$

6. Solve for x:

(a) $2^{2x} + 3.2^{x} - 4 = 0$ (b) $4^{x} - 2^{x-2} = 0$

(c) $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} - 5 = 0$ (d) $3^{x} + 3.3^{-x} - 4 = 0$

7. (a) Factorise: $2\sqrt{x^5} - \sqrt{x^3}$

(b) Hence solve for x if: $2\sqrt{x^5} - \sqrt{x^3} = 2x - 1$

SOME CHALLENGES

1. Show that $(\sqrt{3})^{-2x} + 4.3^{1-x} = \dfrac{13}{3^x}$

2. Show that $\dfrac{9^{x+1} - 6.3^{2x}}{(\sqrt{3})^{4x+1}} = \sqrt{3}$

3. Show that $\left(\dfrac{\sqrt{x^5} - \sqrt{x^3}}{x}\right)^2 = x(x-1)^2, x > 0$

4. Calculate $4.(\sqrt{2})^3.8^{\frac{2}{3}}$ without the use of a calculator (simplest surd form).

5. Given: $P = \dfrac{5^x + 5^x + 5^x + 5^x + 5^x}{5^{x+5}}$. If $x = 2011$ then,

(1) $P = 5^{8039}$ (2) $P = 1$ (3) $P = 5^{-4}$ (4) $P = 5$

Indicate the correct answer and show working out.

6. Solve for x: $2^x - (\sqrt{2})^{x+8} + 64 = 0$

7. Without the use of a calculator determine which number is the largest:

(a) 3^{360} or 2^{540} (b) $\sqrt[5]{16}$ or $\sqrt[3]{5}$

8. Simplify: $\sqrt[n]{4^{2n}.\left(\dfrac{1}{8}\right)^{n+1}.8}$, $n \in N, n \geq 2$

9. Calculate: $\dfrac{2^{2013} - 6.2^{2011}}{4^{1010}}$. Leave the answer in positive exponential form.

10. (a) Show that $2^{2011}.5^{2007} = 1,6 \times 10^{2008}$

(b) Hence determine the sum of the digits of $2^{2011}.5^{2007}$

11. (a) Write each term in exponential form:

(1) $T_1 = \sqrt{x}$ (2) $T_2 = \sqrt{x\sqrt{x}}$ (3) $T_3 = \sqrt{x\sqrt{x\sqrt{x}}}$

(4) $T_4 = \sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$

(b) Identify a pattern and then determine:

(1) T_{10} (2) T_n (3) T_{100} (4) T_∞

12. Solve for x and y if: $9^{x+y} = 3^{y+4}$ and $5x + 4y = 11$

13. Network-marketing is a business model that is quite prevelant in South Africa. In this scheme, people are told how to become rich quickly by using the rules of exponents. Suppose that you get three people to sign up in your business. These three people (in your down-line) must then each get three people to sign up so that they too each have three people in their down-line. This process must continue so that your business grows with hundreds of people in your business. How many people will be in your business in the 15^{th} down-line? What are the problems associated with this kind of business model?

CHAPTER 2 – EQUATIONS AND INEQUALITIES

COMPLETING THE SQUARE

Completing the square is a technique used to rewrite a quadratic expression of the form ax^2+bx+c in the form $a(x+p)^2+q$. The application of completing the square is extremely useful when sketching the graphs of quadratic functions and determining the maximum or minimum values of quadratic expressions. Quadratic equations can also be solved using this technique. However, before discussing the technique of completing the square, it is necessary to first deal with the concept of quadratic expressions which are perfect squares. If the square root of a number works out to be a rational number, then that number is a perfect square. For example, 1; 4; 9; 16; 25; 36 are perfect squares since the square root of each number is a rational number.

The quadratic expressions $(x+2)^2;(x-5)^2;(x+1)^2;4(x-2)^2$ are also perfect squares.

Investigation

(a) Show that the following trinomials are perfect squares:

(1) x^2+2x+1 (2) x^2-6x+9 (3) $x^2+8x+16$

(b) What is the relationship between the last term and the middle term of the trinomials in (a)?

(c) State a rule that connects the last term to the coefficient of the middle term of a perfect square quadratic trinomial.

Solutions

(a) (1) $x^2+2x+1=(x+1)(x+1)=(x+1)^2$

(2) $x^2-6x+9=(x-3)(x-3)=(x-3)^2$

(3) $x^2+8x+16=(x+4)(x+4)=(x+4)^2$

(b) (1) $x^2+2x+1=x^2+2x+(1)^2=x^2+2x+\left(\frac{1}{2}\times 2\right)^2$

Coefficient of middle term $=2$ Last term $=\left(\frac{1}{2}\times 2\right)^2$

(2) $x^2-6x+9=x^2-6x+(-3)^2=x^2-6x+\left(\frac{1}{2}\times -6\right)^2$

Coefficient of middle term $=-6$ Last term $=\left(\frac{1}{2}\times -6\right)^2$

(3) $x^2+8x+16=x^2+8x+(4)^2=x^2+8x+\left(\frac{1}{2}\times 8\right)^2$

Coefficient of middle term $=8$ Last term $=\left(\frac{1}{2}\times 8\right)^2$

The last term of a perfect square trinomial is the square of half of the coefficient of x in the middle term.

$$\text{Last term} = \left(\frac{1}{2}\times(\text{the coefficient of } x)\right)^2$$

(c) In a perfect square trinomial of the form $x^2 + bx + c$, the last term (c) is equal to the square of half the coefficient of x (which is b) in the middle term. The rule is as follows:

In the quadratic trinomial $x^2 + bx + c$: $\quad c = \left(\frac{1}{2}\times b\right)^2$

EXAMPLE 1

Determine the value of c (the last term) in each of the following quadratic expressions if the expressions are perfect squares:

(a) $x^2 + 4x + c$ (b) $x^2 - 10x + c$ (c) $x^2 - 5x + c$

Solutions

(a) $x^2 + 4x + \left(\frac{1}{2}\times 4\right)^2 = x^2 + 4x + (2)^2 = x^2 + 4x + 4$

(b) $x^2 - 10x + \left(\frac{1}{2}\times -10\right)^2 = x^2 - 10x + (-5)^2 = x^2 - 10x + 25$

(c) $x^2 - 5x + \left(\frac{1}{2}\times -5\right)^2 = x^2 - 5x + \left(\frac{-5}{2}\right)^2 = x^2 - 5x + \frac{25}{4}$

EXAMPLE 2

Rewrite the following expressions in the form $a(x+p)^2 + q$ by completing the square.

(a) $x^2 + 2x$ (b) $x^2 - 9x$ (c) $x^2 + 8x + 10$

Solution

(a) $x^2 + 2x$

The first step is to determine $\left(\frac{1}{2}\times(\text{the coefficient of } x)\right)^2$:

$$\left(\frac{1}{2}\times(2)\right)^2 = (1)^2 = 1$$

Now **add the number** $(1)^2$ **to** $x^2 + 2x$ **and then subtract** it so as not to change the value of the expression.

$\therefore x^2 + 2x$

$= x^2 + 2x + (1)^2 - (1)^2$

$= x^2 + 2x + 1 - 1$

$= (x+1)^2 - 1$

$\therefore x^2 + 2x = (x+1)^2 - 1$

(b) $x^2 - 9x$

The first step is to determine $\left(\frac{1}{2}\times\left(\text{the coefficient of } x\right)\right)^2$

$\left(\frac{1}{2}\times -9\right)^2 = \left(-\frac{9}{2}\right)^2 = \frac{81}{4}$

Now add the number $\left(-\frac{9}{2}\right)^2$ to $x^2 - 9x$ and then subtract

$\therefore x^2 - 9x$

$= x^2 - 9x + \left(-\frac{9}{2}\right)^2 - \left(-\frac{9}{2}\right)^2$

$= x^2 - 9x + \frac{81}{4} - \frac{81}{4}$

$= \left(x - \frac{9}{2}\right)^2 - \frac{81}{4}$

$\therefore x^2 - 9x = \left(x - \frac{9}{2}\right)^2 - \frac{81}{4}$

(c) $x^2 + 8x + 10$

In this case we have three terms and not two. When we are completing the square, we will only consider the first two terms and work with them, while the last term will just run along.

$\left(\frac{1}{2}\times\left(\text{the coefficient of } x\right)\right)^2 = \left(\frac{1}{2}\times 8\right)^2 = (4)^2 = 16$

$= x^2 + 8x + (4)^2 - (4)^2 + 10$

$= x^2 + 8x + 16 - 16 + 10$

$= \left(x + 4\right)^2 - 16 + 10$

$= \left(x + 4\right)^2 - 6$

$\therefore x^2 + 8x + 10 = \left(x + 4\right)^2 - 6$

EXERCISE 1

Complete the square for each of the following quadratic expressions.

(a)	$x^2 - 4x$	(b)	$x^2 + 5x$	(c)	$x^2 - 6x$
(d)	$x^2 - 6x + 8$	(e)	$x^2 - 8x + 10$	(f)	$x^2 + x + 1$
(g)	$x^2 + 3x - 8$	(h)	$x^2 - 10x - 2$		

All of the expressions we have dealt with thus far have had 1 as a coefficient of the term in x^2.

Completing the square in trinomials where the coefficient of x^2 is not 1 requires an important first step.

EXAMPLE 3

Rewrite $2x^2 - 7x + 6$ in the form $a(x + p)^2 + q$ by completing the square.

Solution

$2x^2 - 7x + 6$ "Take out" the coefficient of x^2 for the first two terms. This is necessary so that you can work with an expression with a coefficient of 1 for x^2.

$= 2\left(x^2 - \frac{7}{2}x\right) + 6$

It is now possible to complete the square for the expression inside the bracket.

$$\left(\frac{1}{2} \times (\text{the coefficient of } x)\right)^2 = \left(\frac{1}{2} \times -\frac{7}{2}\right)^2 = \left(-\frac{7}{4}\right)^2 = \frac{49}{16}$$

$\therefore 2\left(x^2 - \frac{7}{2}x\right) + 6$

$= 2\left(x^2 - \frac{7}{2}x + \left(-\frac{7}{4}\right)^2 - \left(-\frac{7}{4}\right)^2\right) + 6$

$= 2\left(x^2 - \frac{7}{2}x + \frac{49}{16} - \frac{49}{16}\right) + 6$ Add and subtract $\left(\frac{1}{2} \times (\text{the coefficient of } x)\right)^2$

$= 2\left(\left(x - \frac{7}{4}\right)^2 - \frac{49}{16}\right) + 6$ Factorise the perfect square trinomial

$= 2\left(x - \frac{7}{4}\right)^2 - \frac{49}{8} + 6$ Multiply 2 into the brackets

$= 2\left(x - \frac{7}{4}\right)^2 - \frac{1}{8}$

Summary of the basic procedure for <u>completing the square</u>:

1. "Take out" the coefficient of x^2 for the first two terms if necessary.
2. Add and subtract $\left(\frac{1}{2} \times (\text{the coefficient of } x)\right)^2$
3. Factorise the perfect square trinomial and multiply.

EXERCISE 2

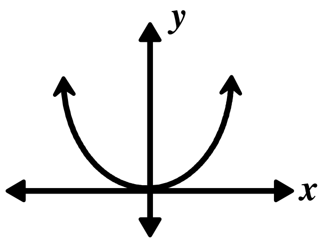
1. Complete the square for each of the following:
   (a) $2x^2 - 8x + 6$ (b) $3x^2 - 6x + 6$
   (c) $-x^2 - 2x - 2$ (d) $2x^2 - 5x + 4$
2. Show by completing the square that $2x^2 + 6x + 10 = 2\left(x + \frac{3}{2}\right)^2 + 5\frac{1}{2}$
3. Show by completing the square that $-x^2 + 4x - 9 = -(x - 2)^2 - 5$

Determining the maximum and minimum values of quadratic expressions

Completing the square is an effective technique for determining the maximum or minimum value of a quadratic expression.

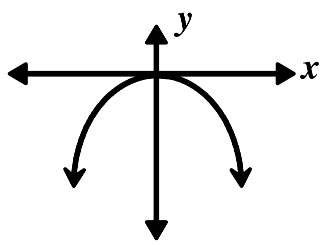
Consider the graph of $y = x^2$.

The expression x^2 has a **minimum value of 0**.
It has a "floor" at zero but no "ceiling" on top. It doesn't go lower than zero on the *y*-axis.



Consider the graph of $y = -x^2$.

The expression x^2 has a **maximum value of 0**.
It has a "ceiling" at zero but no "floor" at the bottom. It doesn't go higher than zero on the *y*-axis.



Some important principles can be deduced from the expression $a(x+p)^2$:

If $a > 0$, the expression $a(x+p)^2$ will have **minimum value of 0**

If $a < 0$, the expression $a(x+p)^2$ will have **maximum value of 0**

EXAMPLE 4

Determine the maximum or minimum values of the following expressions:

(a)	x^2+4	(b)	$-x^2+3$	(c)	$-2x^2-5$	(d)	$3x^2-9$
(e)	$(x-1)^2$	(f)	$-2(x-1)^2+8$	(g)	$5(-x+9)^2-2$		

Solutions

(a)	x^2+4	The coefficient of x^2 is $+1$ and therefore there is a minimum value which is 4 (4 **more** than 0)
(b)	$-x^2+3$	The coefficient of x^2 is -1 and therefore there is a maximum value which is 3 (3 **more** than 0)
(c)	$-2x^2-5$	The coefficient of x^2 is -2 and therefore there is a maximum value which is -5 (5 **less** than 0)
(d)	$3x^2-9$	The coefficient of x^2 is $+3$ and therefore there is a minimum value which is -9 (9 **less** than 0)
(e)	$(x-1)^2$	The coefficient of $(x-1)^2$ is $+1$ and therefore there is a minimum value which is 0
(f)	$-2(x-1)^2+8$	The coefficient of $(x-1)^2$ is -2 and therefore there is a maximum value which is 8 (8 **more** than 0)
(g)	$5(-x+9)^2-2$	The coefficient of $(-x+9)^2$ is $+5$ and therefore there is a minimum value which is -2 (2 **less** than 0)

EXAMPLE 5

(a) Show that $x^2 - 6x + 16$ has a minimum value of 7 by completing the square.

Solution

$\left(\frac{1}{2} \times \left(\text{the coefficient of } x\right)\right)^2 = \left(\frac{1}{2} \times -6\right) = (-3)^2 = 9$

$\therefore x^2 - 6x + 16$

$= x^2 - 6x + (-3)^2 - (-3)^2 + 16$

$= x^2 - 6x + 9 - 9 + 16$

$= (x-3)^2 - 9 + 16$

$= (x-3)^2 + 7$

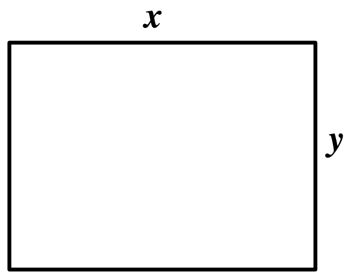
$\therefore$ The minimum value is 7

(b) A farmer wants to fence off his cattle within a rectangular area. He has 18km of fencing.

(1) Show that $y = 9 - x$

(2) Write a formula for the area of the farm in terms of x.

(3) What is the largest rectangular area that can be enclosed with the 18km of fencing?



Solutions

(1) Perimeter $= 2l + 2b$

$\therefore$ Perimeter $= 2x + 2y$

$\therefore 2x + 2y = 18$

$\therefore x + y = 9$

$\therefore y = 9 - x$

(2) Area $= l \times b$

$\therefore$ Area $= x \times y = xy$

$\therefore$ Area $= x(9 - x) = 9x - x^2$

(3) Complete the square in order to determine the maximum area.

Area $= 9x - x^2 = -x^2 + 9x$

$\therefore$ Area $= -(x^2 - 9x)$

$\therefore$ Area $= -\left(x^2 - 9x + \left(-\frac{9}{2}\right)^2 - \left(-\frac{9}{2}\right)^2\right)$

$\therefore$ Area $= -\left(x^2 - 9x + \frac{81}{4} - \frac{81}{4}\right)$

$\therefore$ Area $= -\left(\left(x - \frac{9}{2}\right)^2 - \frac{81}{4}\right)$

$\therefore$ Area $= -\left(x - \frac{9}{2}\right)^2 + \frac{81}{4}$

$\therefore$ The maximum area that can be enclosed is $\frac{81}{4}\text{km}^2$

EXERCISE 3

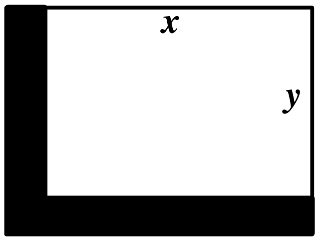
1. Determine whether each of the following expressions have a maximum or a minimum and then find that value.
   (a) $-9x^2$ (b) $9x^2$ (c) $-(2-x)^2$
   (d) x^2+1 (e) $4(3x+9)^2$ (f) $-2(x-3)^2+9$
   (g) $2(-x-1)^2-3$

2. Determine the maximum value of $-x^2+8x-12$ by completing the square.

3. Determine the minimum value of $2x^2+6x$ by completing the square.

4. Show that x^2+x+1 is always positive for all real values of x.

5. A farmer has 100 metres of wire fencing from which to build a rectangular chicken run. He intends using two adjacent walls for two sides of the rectangular enclosure.
   (a) Determine a formula for the enclosed area in terms of x.
   (b) Determine the dimensions which will give a maximum enclosed area.



QUADRATIC EQUATIONS

A quadratic equation has the form $ax^2+bx+c=0$ and has at most two real solutions (roots). In Grade 10, quadratic equations were solved by factorising the quadratic expression and then applying the zero factor law, which states that if $ab=0$, then either $a=0$ or $b=0$.
In Grade 11, two other methods of solving quadratic equations will be discussed:
- Completing the square
- Quadratic formula

EXAMPLE 6 **(Revision of Grade 10 work)**

Solve each of the following equations:

(a) $x^2+5x=6$
The first step is to always write the equation in standard quadratic equation form.

$\therefore x^2+5x-6=0$			
$\therefore (x+6)(x-1)=0$			Factorise the left side
$\therefore x+6=0$	or	$x-1=0$	Apply the zero factor law
$\therefore x=-6$	or	$x=1$	

(b) $x^2=4x$

$\therefore x^2-4x=0$			Write the equation in standard form
$\therefore x(x-4)=0$			Factorise the left side
$\therefore x=0$	or	$x-4=0$	Apply the zero factor law
		$\therefore x=4$	

(c) $-4x^2 = 10x - 6$

$\therefore -4x^2 - 10x + 6 = 0$ — Write the equation in standard form

$\therefore 2x^2 + 5x - 3 = 0$ — Divide both sides by -2.

$\therefore (2x - 1)(x + 3) = 0$ — Factorise the left side

$\therefore 2x - 1 = 0$ or $x + 3 = 0$ — Apply the zero factor law

$\therefore 2x = 1$ or $x = -3$

$\therefore x = \frac{1}{2}$

(d) $x^2 = 25$

There are two methods of solving this equation:

Method 1	**Method 2**
$x^2 - 25 = 0$	$x^2 - 25 = 0$
$\therefore (x+5)(x-5) = 0$	$\therefore x^2 = 25$
$\therefore x = -5$ or $x = 5$	$\therefore x = \pm\sqrt{25}$
$\therefore x = \pm 5$	$\therefore x = \pm 5$

The second method is useful in solving any equation of the form $x^2 = a$.

Consider the following examples:

(e) $x^2 = 5$

$\therefore x = \pm\sqrt{5}$

Irrational solution

(f) $x^2 = -5$

$\therefore x = \pm\sqrt{-5}$

Non-real solution

EXERCISE 4

Solve the following equations:

(a)	$x^2 - 5x = 6$	(b)	$x^2 = x$	(c)	$-x^2 - 13x = 12$
(d)	$4x^2 = 36x - 32$	(e)	$(x-2)(x+3) = 14$	(f)	$(2x+1)(3-x) = 6$
(g)	$x(x-1) = 2$	(h)	$x^2 - 16 = 0$	(i)	$x^2 = 9$
(j)	$x^2 - 6 = 0$	(k)	$x^2 - 7 = 0$	(l)	$x^2 + 7 = 0$

SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

EXAMPLE 7

Solve the following equations by completing the square:

(a) $x^2 - 4x - 5 = 0$ (b) $2x^2 + 10x - 14 = 0$

Solutions

(a) $x^2 - 4x = 5$ — Move the constant to the RHS

Now determine $\left(\frac{1}{2} \times (\text{the coefficient of } x)\right)^2$

$\therefore \left(\frac{1}{2} \times -4\right)^2 = (-2)^2 = 4$

$\therefore x^2 - 4x + (-2)^2 - (-2)^2 = 5$ Add and subtract $(-2)^2$

$\therefore x^2 - 4x + 4 - 4 = 5$

$\therefore x^2 - 4x + 4 = 5 + 4$ Isolate the perfect square trinomial

$\therefore (x-2)^2 = 9$ Factorise LHS and simplify RHS

$\therefore x - 2 = \pm\sqrt{9}$ Square root both sides (NB: $\pm$)

$\therefore x - 2 = 3$ or $x - 2 = -3$ Do the split and solve.

$\therefore x = 5$ or $x = -1$

(b) $2x^2 + 10x - 14 = 0$

$\therefore 2x^2 + 10x = 14$ Move the constant to the right

$\therefore x^2 + 5x = 7$ Divide by 2

$\therefore x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 = 7$ Add and subtract $\left(\frac{5}{2}\right)^2$

$\therefore x^2 + 5x + \frac{25}{4} - \frac{25}{4} = 7$

$\therefore x^2 + 5x + \frac{25}{4} = 7 + \frac{25}{4}$ Isolate the perfect square trinomial

$\therefore \left(x + \frac{5}{2}\right)^2 = \frac{53}{4}$ Factorise LHS and simplify RHS

$\therefore x + \frac{5}{2} = \pm\sqrt{\frac{53}{4}}$ Square root both sides (NB: $\pm$)

$\therefore x + \frac{5}{2} = \frac{\sqrt{53}}{2}$ or $x + \frac{5}{2} = \frac{-\sqrt{53}}{2}$ Do the split and solve.

$\therefore x = \frac{-5 + \sqrt{53}}{2}$ or $\therefore x = \frac{-5 - \sqrt{53}}{2}$

Alternatively you can add $\left(\frac{1}{2} \times (\text{the coefficient of } x)\right)^2$ to both sides instead of adding and subtracting it on the one side.

$\therefore x^2 + 5x = 7$

$\therefore x^2 + 5x + \left(\frac{5}{2}\right)^2 = 7 + \left(\frac{5}{2}\right)^2$. The equality still holds true because the same value is added to both sides of the equation. Then proceed from there.

The CAB principle: Refer to the standard form: $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c} = 0$ Step 1: "Move" the **c-value** across Step2: Divide both sides by the **a-value** Step 3: Add $\left(\frac{1}{2} \times \textbf{b-value}\right)^2$ to both sides of the equation Step 4: Factorise and solve the equation by square-rooting both sides

EXERCISE 5

1. Solve for x by completing the square leaving answers in simplest surd form.
   (a) $x^2 + 4x - 6 = 0$ (b) $x^2 + x - 1 = 0$ (c) $x^2 - 5x + 2 = 0$
   (d) $2x^2 + 3x - 5 = 0$ (e) $3x^2 - 7x + 3 = 0$ (f) $-x^2 + 6x + 8 = 0$
   (g) $2x^2 - 5x = 10$

2. Show by completing the square that the solutions to the equation $ax^2 - bx - a = b$ are $x = -1$ and $x = \dfrac{a+b}{a}$.

3. Given the quadratic equation in x: $x^2 + x + 1 = 0$
   Show by completing the square that the equation has no real solutions.

*4. By completing the square, show that the solutions of any quadratic equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$ can be determined by using the formula:
   $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (This is called the quadratic formula)

SOLVING QUADRATIC EQUATIONS BY USING THE QUADRATIC FORMULA

The solutions of any quadratic equation in standard form $ax^2 + bx + c = 0$ where $a \neq 0$ can be determined using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is called the quadratic formula.

EXAMPLE 8

Solve for x:

(a) $x^2 + 9x = 36$ (b) $-3x^2 = 7x - 12$ (c) $x^2 + x + 2 = 0$

Solutions

(a) $x^2 + 9x = 36$

$\therefore x^2 + 9x - 36 = 0$ The first step is to write the equation in standard quadratic equation form.

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ State the formula and find a, b and c.

$a = 1$
$b = 9$
$c = -36$

$$\therefore x = \frac{-(9) \pm \sqrt{(9)^2 - 4(1)(-36)}}{2(1)}$$ Substitute the values of a, b and c.

$$\therefore x = \frac{-9 \pm \sqrt{225}}{2}$$

$$\therefore x = \frac{-9+15}{2} \quad \text{or} \quad x = \frac{-9-15}{2}$$ Do the split and solve

$$\therefore x = 3 \quad \text{or} \quad x = -12$$

Note: This equation could have been solved in the usual way:

$x^2 + 9x - 36 = 0$

$\therefore (x+12)(x-3) = 0$

$\therefore x = -12$ or $x = 3$

(b) $-3x^2 = 7x - 12$

$\therefore -3x^2 - 7x + 12 = 0$ Write the equation in the standard form

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-3)(12)}}{2(-3)}$$

$a = -3$
$b = -7$
$c = 12$

$$\therefore x = \frac{7 \pm \sqrt{193}}{-6}$$ The answer is in surd form

If you are required to round off to two decimal digits, continue as follows:

$$\therefore x = \frac{7+\sqrt{193}}{-6} \quad \text{or} \quad x = \frac{7-\sqrt{193}}{-6}$$

$$\therefore x = -3{,}48 \quad \text{or} \quad x = 1{,}15$$

(c) $x^2 + x + 2 = 0$ Equation is in standard form

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)}$$

$a = 1$
$b = 1$
$c = 2$

$$\therefore x = \frac{-1 \pm \sqrt{-7}}{2}$$

$\therefore$ Non-real solution

EXERCISE 6

1. Solve the following equations by using the quadratic formula. Leave your answers in surd form, where appropriate.

(a)	$x^2 - 7x = 9$	(b)	$3x^2 + 2 = 9x$
(c)	$2x^2 - x - 1 = 0$	(d)	$(2x-1)(x+3) = 15$
(e)	$3x - 2x(x+1) = 2$	(f)	$x(x+2) + 5 = 0$

2. Solve the following equations by using the quadratic formula. Leave your answers rounded off to one decimal place.

(a)	$x^2 = 3x - 9$	(b)	$-x^2 = 2 - 7x$
(c)	$x^2 - x + 1 = 0$	(d)	$(x-1)(3x+2) = 11$
(e)	$x^2 + 2 = 8x(x-2)$		

EQUATIONS WITH FRACTIONS

EXAMPLE 9

Solve for x:

(a) $\dfrac{10}{x} + \dfrac{3x}{x-2} = 7$ (b) $\dfrac{5x+9}{x^2-2x-3} - \dfrac{2x}{3-x} = \dfrac{x}{x+1}$

Solutions

The first step is to find the LCD and to state the restrictions. The restrictions are the value(s) of x for which the denominator(s) are equal to zero.
Remember that division by 0 is undefined.

(a) $\dfrac{10}{x} + \dfrac{3x}{x-2} = 7$

LCD: $x(x-2)$ and **RESTRICTIONS:** $x \neq 0$ and $x - 2 \neq 0$
$\therefore x \neq 2$

Multiply each term by the LCD: $x(x-2)$

$$\therefore \frac{10}{x} \times \frac{x(x-2)}{1} + \frac{3x}{x-2} \times \frac{x(x-2)}{1} = \frac{7}{1} \times \frac{x(x-2)}{1}$$

$\therefore 10(x-2) + 3x(x) = 7x(x-2)$

$\therefore 10x - 20 + 3x^2 = 7x^2 - 14x$

$\therefore 0 = 4x^2 - 24x + 20$ Write equation in standard form

$\therefore 0 = x^2 - 6x + 5$ Divide by the numerical factor 4

$\therefore 0 = (x-5)(x-1)$ Factorise

$\therefore x = 5$ or $x = 1$ **NB!! Check answers with restrictions**.

Both solutions are valid.

(b) $\frac{5x+9}{x^2-2x-3}-\frac{2x}{3-x}=\frac{x}{x+1}$

$\therefore \frac{(5x+9)}{(x-3)(x+1)}-\frac{2x}{(3-x)}=\frac{x}{(x+1)}$ Factorise the denominator

$\therefore \frac{(5x+9)}{(x-3)(x+1)}-\frac{2x}{-(x-3)}=\frac{x}{x+1}$ Change in sign rule: $3-x=-(x-3)$

$\therefore \frac{(5x+9)}{(x-3)(x+1)}+\frac{2x}{(x-3)}=\frac{x}{(x+1)}$.

LCD: $(x-3)(x+1)$ and **RESTRICTIONS:** $x \neq 3$ and $x \neq -1$

Multiply each term by the LCD: $(x-3)(x+1)$

$\therefore 5x+9+2x(x+1)=x(x-3)$

$\therefore 5x+9+2x^2+2x=x^2-3x$

$\therefore 2x^2+7x+9=x^2-3x$

$\therefore x^2+10x+9=0$

$\therefore (x+9)(x+1)=0$

$\therefore x=-9$ or $x=-1$

But $x \neq -1$

$\therefore x=-9$

Special cases

$0=0 \Rightarrow$ Infinite many solutions	$0=$ number $\Rightarrow$ No solution
Example:	**Example**:
Solve for x if:	Solve for x if:
$3x+6=3(x+2)$	$3x+8=3(x+2)$
$\therefore 3x+6=3x+6$	$\therefore 3x+8=3x+6$
$\therefore 0=0$	$\therefore 2=0$
$\therefore x \in$ Real numbers	No solution

EXERCISE 7

Solve for x leaving answers rounded off to one decimal place where necessary:

(a) $x-3=\frac{18}{x}$ (b) $\frac{x-2}{x-1}-\frac{5}{x+2}=\frac{7}{x-1}$

(c) $\frac{x+6}{x^2-4}-\frac{2}{x-2}=\frac{-1}{x+2}$ (d) $\frac{x^2-1}{x+1}=-2$

(e) $\frac{6x}{x-3}-\frac{x-3}{x+3}=\frac{9}{9-x^2}$ (f) $\frac{x}{x-2}=\frac{1}{x-3}-\frac{2}{2-x}$

(g) $x+1=\frac{15}{x-1}$ (h) $\frac{3x+4}{x+6}=\frac{3x-2}{x-3}-\frac{21x}{x^2+3x-18}$

(i) $\frac{6}{x^2+5x}=\frac{21-x}{5x}-\frac{18}{5+x}$ (j) $x-\frac{1}{x}=1$

THE NATURE OF ROOTS OF A QUADRATIC EQUATION

The **roots** of an equation refer to the **solutions** of that equation. When solving a quadratic equation one is also finding the ***x*-intercepts** of a quadratic function that are represented by that equation (see Chapter 5).
The roots of a quadratic equation can be:

- **Real or non-real**
- **Rational or irrational**
- **Equal (two equal solutions) or unequal (two different solutions)**

Note:

1. $\sqrt{\text{negative number}}$ is a non-real number
   Examples: $\sqrt{-25}$, $\sqrt{-3}$, $\sqrt{-100}$ etc.
2. $\sqrt{\text{perfect square}}$ is a real, rational number
   Examples: $\sqrt{25} = 5$, $\sqrt{100} = 10$, $\sqrt{\frac{9}{4}} = \frac{3}{2}$ etc.
3. $\sqrt{\text{not a perfect square}}$ is a real, irrational number
   Examples: $\sqrt{3} = 1{,}732...$, $\sqrt{15} = 3{,}8729....$ etc.

When solving a quadratic equation, the quadratic formula can be used:
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The value of the expression $b^2 - 4ac$ (called delta or Δ), establishes the nature of the roots of the quadratic equation.

Consider the following scenarios. Each equation will be solved by using the quadratic formula. The nature of the roots (solutions) of each equation will be discussed by considering the actual solutions as well as the value of the expression $b^2 - 4ac$.

(a) **Scenario 1: Roots which are non-real**

$$x^2 + 3x + 4 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\therefore x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(4)}}{2(1)}$$
$$\therefore x = \frac{-3 \pm \sqrt{-7}}{2}$$

Therefore the roots are non-real.

Notice that in this expression $b^2 - 4ac = -7$, which is negative. It can be concluded that the roots are non-real since $b^2 - 4ac < 0$ (delta is negative).

(b) **Scenario 2: Roots which are real, rational and unequal**

$x^2 - 5x - 6 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)}$

$\therefore x = \dfrac{5 \pm \sqrt{49}}{2}$

$\therefore x = \dfrac{5 \pm 7}{2}$

$\therefore x = 6 \quad \text{or} \quad x = -1$

Therefore the roots are real, rational and unequal.

Let us consider $b^2 - 4ac$:

$b^2 - 4ac = 49$.

49 is **positive and a perfect square**.

Therefore the roots will be:

- Real because $b^2 - 4ac > 0$ (delta is positive)
- Rational because $b^2 - 4ac =$ perfect square
- Unequal because $b^2 - 4ac > 0$ (delta is positive)

(c) **Scenario 3: Roots which are real, irrational and unequal**

$2x^2 + 3x - 7 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \dfrac{-(3) \pm \sqrt{(3)^2 - 4(2)(-7)}}{2(2)}$

$\therefore x = \dfrac{-3 \pm \sqrt{65}}{4}$

$\therefore x = 1{,}265... \quad \text{or} \quad x = -2{,}765...$

Therefore the roots are real, irrational and unequal.

Let us consider $b^2 - 4ac$:

$b^2 - 4ac = 65$.

65 is **positive and not a perfect square**.

Therefore the roots will be:

- Real because $b^2 - 4ac > 0$ (delta is positive)
- Irrational because $b^2 - 4ac =$ not perfect square
- Unequal because $b^2 - 4ac > 0$ (delta is positive)

(d) **Scenario 4: Roots which are real, rational and equal**

$x^2 - 6x + 9 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)}$

$\therefore x = \dfrac{6 \pm \sqrt{0}}{2}$

$\therefore x = \dfrac{6+0}{2}$ or $\dfrac{6-0}{2}$

$\therefore x = 3$ or $x = 3$

The equation has two equal roots.
Therefore the roots are real, rational and equal.

Let us consider $b^2 - 4ac$:
$b^2 - 4ac = 0$.
0 is **not negative and a perfect square**.
Therefore the roots will be:

- Real because $b^2 - 4ac \geq 0$ (delta is not negative)
- Rational because $b^2 - 4ac = 0$ (perfect square)
- Equal because $b^2 - 4ac = 0$ (delta equals zero)

Determining the nature of the roots of a quadratic equation by considering $\Delta = b^2 - 4ac$ is summarized in the following table:

$b^2 - 4ac$	Roots
$\Delta < 0$	Non-real
$\Delta \geq 0$	Real
$\Delta > 0$ and Δ = perfect square	Real, rational and unequal
$\Delta > 0$ and $\Delta \neq$ a perfect square	Real, irrational and unequal
$\Delta = 0$	Real, rational and equal

<u>EXAMPLE 10</u>

(a) Without solving the equation $3x^2 + 8x - 2 = 0$, discuss the nature of its roots.

<u>Solution</u>

$3x^2 + 8x - 2 = 0$

$\Delta = b^2 - 4ac = (8)^2 - 4(3)(-2) = 88$

$\therefore \Delta = 88$

$\therefore \Delta > 0$ and a not a perfect square

$\therefore$ Roots are real, irrational and unequal.

(b) Show that the roots of the equation $x(6x-7m)=5m^2$ are real, rational and unequal if $m \neq 0$

Solution

$x(6x-7m)=5m^2$

$\therefore 6x^2-7mx=5m^2$

$\therefore 6x^2-7mx-5m^2=0$

$a=6 \qquad b=-7m \qquad c=-5m^2$

$\Delta=b^2-4ac$

$\therefore \Delta=(-7m)^2-4(6)(-5m^2)$

$\therefore \Delta=49m^2+120m^2$

$\therefore \Delta=169m^2$

$\therefore \Delta > 0$ and is a perfect square

$\therefore$ The roots are real, rational and unequal

(c) If $p(x^2+x)=-9$ has equal roots ($p \neq 0$), determine the value(s) of p.

Solution

$p(x^2+x)=-9$

$\therefore px^2+px=-9$

$\therefore px^2+px+9=0$

$\Delta=b^2-4ac$

$\therefore \Delta=p^2-4(p)(9)$

$\therefore \Delta=p^2-36p$

$\therefore$ For the equation to have equal roots (solutions), Δ has to equal zero

$\therefore p^2-36p=0$

$\therefore p(p-36)=0$

$\therefore p=0$ or $p=36$

But $p \neq 0$

$\therefore p=36$

EXERCISE 8

1. Without solving each of the following equations, discuss the nature of the roots:

   (a) $x^2+x+1=0$ (b) $x^2=2(x+1)$ (c) $x^2=4x$

   (d) $-2x^2-16x-32=0$ (e) $(x-2)(2x-1)=5$ (f) $3x+7=\frac{5}{x}$

2. Show that the roots of the equation $mx(x-4)=-4m$ are equal for all real values of m.

3. Show that the roots of the equation $x(x-3m)=-5m^2$ are non-real if $m \neq 0$.

4. (a) Show that the roots of the equation $2x^2 + a(x - a) = 0$ are rational for any real value of a where $a \neq 0$
   (b) Discuss the nature of the roots if $a = 0$.
5. Show that the roots of the equation $x^2 + (p-2)x + (1-p) = 0$ are rational for any real value of p.
6. Show that the roots of the equation $(r+1)x^2 + 4x + 1 = r$ are real for all real values of r.
7. For which value(s) of k will the equation $x^2 - 5x - k = 0$ have:
   (a) equal roots (b) real roots (c) non-real roots
8. For which value(s) of k will the equation $kx^2 + 2kx = -3$ have equal roots where $k \neq 0$
9. For which values of p will the equation $x(4x+3) = -p$ have:
   (a) real roots (b) non-real roots
10. For which values of k will the equation $x^2 + 2kx + 4x + 9k = 0$ have equal roots?
11. For which values of k will the equation $x^2 + x + 2 = 3x - k$ have non-real roots?
12. Calculate the value(s) of m for which the equation $x^2 - 3x + 5m = 0$ has:
    (a) one root (solution) with a value of -2
    (b) equal roots (c) real roots
13. Given: $3(x+1) = x^2 + k$
    (a) For which value(s) of k will the roots of this equation be real?
    (b) Find one value of k so that the roots will be rational.

SIMULTANEOUS EQUATIONS

REVISION OF SIMULTANEOUS LINEAR EQUATIONS (GRADE 10)

EXAMPLE 11

Solve for x and y in the following set of equations.
$2x + 3y = -2$ and $x - 2y = 6$

Label each equation as follows:
$2x + 3y = -2$ …..A
$x - 2y = 6$ ……..B
Multiply each term of equation B by -2. This will help to eliminate the terms in x because they will now differ in sign.
$2x + 3y = -2$ ……..A
$-2x + 4y = -12$ ….C
Now add A and C. (Adding the like terms of the equations)
$2x + 3y = -2$ ………A
$-2x + 4y = -12$........C

$\therefore 7y = -14$
$\therefore y = -2$

(The terms in x are different in sign and will be eliminated when adding).

$2x + 3y = -2$

$\therefore 2x + 3(-2) = -2$ Substitute the y-value into any one of the two given

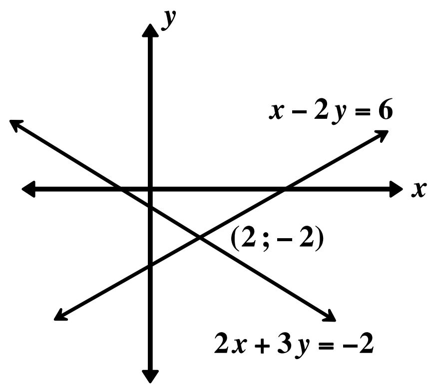
$\therefore 2x - 6 = -2$ equations (equation A was chosen)

$\therefore 2x = 4$

$\therefore x = 2$

Note:

The coordinates of the point of intersection of the lines $2x + 3y = -2$ and $x - 2y = 6$ are obtained by means of solving simultaneous equations.
This will be discussed further in Chapter 5.



OTHER TYPES OF SIMULTANEOUS EQUATIONS

The following examples deal with simultaneous equations involving linear and non-linear equations.

EXAMPLE 12

Solve for x and y simultaneously: $y = x^2 - 1$ and $y - x = 5$

Solution

In this example we are dealing with a linear equation $y - x = 5$ and a non-linear, quadratic equation $y = x^2 - 1$. The method is as follows:

Make y the subject of the formula in the linear equation:

$y - x = 5$

$\therefore y = x + 5$

Now substitute $x + 5$ for y in the quadratic equation and then solve for x:

$y = x^2 - 1$

$\therefore x + 5 = x^2 - 1$

$\therefore 0 = x^2 - x - 6$

$\therefore 0 = (x - 3)(x + 2)$

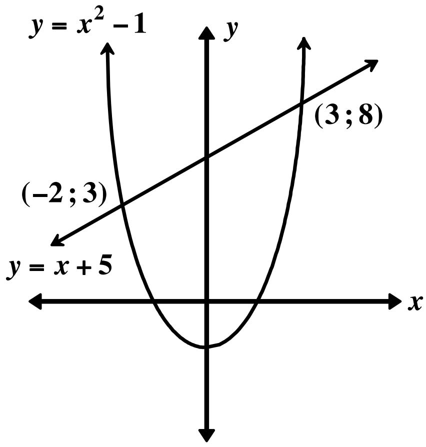
$\therefore x = 3$ or $x = -2$

The corresponding values of y can be determined by substituting the values for x in the linear equation $y = x + 5$:

For $x = 3$	For $x = -2$
$y = 3 + 5 = 8$	$y = -2 + 5 = 3$

The solution can be written in coordinate form as follows: $(3 ; 8)$ and $(-2 ; 3)$

The graphs of the two functions are represented in the diagram alongside. The solution represents the coordinates of the points of intersection of the two graphs.
This will be discussed further in Chapter 5.



EXAMPLE 13

(a) Solve for x and y simultaneously:
$2x - y = 1$ and $x^2 - yx = 3x - 3$

Solution

Label each equation as follows:

$2x - y = 1$……A and $x^2 - yx = 3x - 3$ ……B

Now choose either one of the equations and solve for one of the variables (preferably with a coefficient of 1). Let's solve for the variable y in equation A.

$2x - y = 1$

$-y = -2x + 1$

$\therefore y = 2x - 1$…...C

Replace the variable y in equation B with $2x - 1$ and solve for x.

$\therefore x^2 - (2x - 1)x = 3x - 3$

$\therefore x^2 - 2x^2 + x = 3x - 3$

$\therefore 0 = x^2 + 2x - 3$

$\therefore 0 = (x + 3)(x - 1)$

$\therefore x = -3$ or $x = 1$

Now substitute $x = -3$ and $x = 1$ into either equation A, B or C to get y. Clearly equation C is the best option because y is the subject of that equation.

If $x = -3$ then $y = 2(-3) - 1 = -7$

If $x = 1$ then $y = 2(1) - 1 = 1$

(b) Determine the values of x and y which satisfy the following system of equations: $x + 2y - 3 = 0$ and $y - 3x = -x^2 + 6$

Solution

Let $x + 2y - 3 = 0$……A and $y - 3x = -x^2 + 6$……B

Now choose either one of the equations and solve for one of the variables (preferably with a coefficient of 1). Let's solve for the variable y in equation B

$y - 3x = -x^2 + 6$

$\therefore y = -x^2 + 3x + 6$ ……C

Replace the variable y in equation A with $-x^2 + 3x + 6$ and solve for x.

$x+2(-x^2+3x+6)-3=0$

$x-2x^2+6x+12-3=0$

$\therefore 0=2x^2-7x-9$

$\therefore 0=(2x-9)(x+1)$ (or use the quadratic formula)

$\therefore 2x-9=0$ or $x+1=0$

$\therefore 2x=9$ or $x=-1$

$\therefore x=\frac{9}{2}$

Now substitute $x=\frac{9}{2}$ and $x=-1$ into equation C to obtain the value of y.

If $x=\frac{9}{2}$ then $y=-\left(\frac{9}{2}\right)^2+3\left(\frac{9}{2}\right)+6=-\frac{3}{4}$

If $x=-1$ then $y=-(-1)^2+3(-1)+6=2$

(c) Determine the coordinates of the points of intersection of the graphs:

$y=\frac{-4}{x+3}+9$ and $y=-\frac{3}{2}x+2$

Solution

Both equations are written in the form $y=.....$ which makes solving these simultaneous equations a whole lot easier. We may therefore proceed as follows:

$y=y$

$\therefore \frac{-4}{x+3}+9=-\frac{3}{2}x+2$ LCD: $2(x+3)$

$\therefore \frac{-4}{(x+3)}\times 2(x+3)+9\times 2(x+3)=-\frac{3x}{2}\times 2(x+3)+2\times 2(x+3)$

$\therefore -4(2)+9(2)(x+3)=-3x(x+3)+2(2)(x+3)$

$\therefore -8+18x+54=-3x^2-9x+4x+12$

$\therefore 18x+46=-3x^2-5x+12$

$\therefore 3x^2+23x+34=0$

$\therefore (3x+17)(x+2)=0$ (or use the quadratic formula)

$\therefore 3x=-17$ or $x=-2$

$\therefore x=\frac{-17}{3}$

If $x=-\frac{17}{3}$ then $y=-\frac{3}{2}\left(-\frac{17}{3}\right)+2=\frac{21}{2}$

If $x=-2$ then $y=-\frac{3}{2}(-2)+2=5$

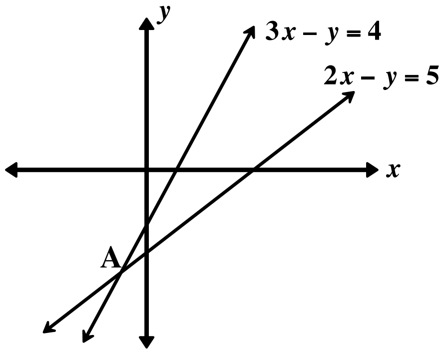
EXERCISE 9

1. Determine the value of x and y which satisfy the following systems of linear equations:

   (a) $x - y = 2$ and $2x + y = 10$ (b) $y - 3x = -2$ and $7x - 2y = 8$

   (c) $3x + 5y = 8$ and $x - 2y = -1$ (d) $7x - 3y = 41$ and $3x - y = 17$

2. Two straight lines with equations $3x - y = 4$ and $2x - y = 5$ are represented in the diagram alongside.
   Determine the coordinates of A, the point of intersection of the lines.



3. Solve for x and y in each of the following sets of equations.

   (a) $y - 2x = 2$ and $x^2 - 2x = 3 - y$

   (b) $y = -x - 3$ and $y = 2x^2 - 3x - 3$

   (c) $3x - y = 2$ and $3y + 9x^2 = 4$

   (d) $2y - x = 2$ and $4y - 2x^2 = x - 4$ (to one decimal place)

   (e) $3x = y + 4$ and $y^2 - xy = 9x + 7$

   (f) $2y + 3x = 7$ and $y = x^2 - 3x + 1$

   (g) $x + 2y = 0$ and $y - xy = \frac{1}{2}x^2 - 2$

   (h) $y = \frac{-6}{x+2} - 1$ and $y + 2x - 6 = 0$

   (i) $y = \frac{3}{x} + 3$ and $3y - x = 1$

4. Given: $a^2 - 5ab + 4b^2 = 0$

   (a) Solve for a in terms of b.

   (b) Hence or otherwise determine the values(s) of the ratio $\frac{a}{b}$

   (c) If the sum of a and b is 2, determine the values of a and b.

SOLVING WORD PROBLEMS

Solving word problems requires you to translate words into mathematical statements. The table below contains typical examples of words translated into mathematical statements.

IN ENGLISH	IN MATHEMATICAL TERMS
The sum of two numbers	$a + b$
The difference between numbers	$a - b$

The product of two numbers	$a \times b = ab$
The quotient of two numbers	$a \div b$
A number is increased by 7	$a + 7$
A number is decreased by 7	$a - 7$
Four times a number	$4a$
A number 5 more than a	$a + 5$
A number 3 less than a	$a - 3$
A number's square	a^2
The sum of three consecutive numbers	$a - 1 + a + a + 1$
An even number	$2a$
An odd number	$2a - 1$ or $2a + 1$

A few things to consider when attempting word problems

- Read through the problem a few times and highlight key numbers and information, especially comparisons.
- Let the quantity you are required to determine be equal to x (or any other letter).
- Make sure that you work with the same units. For example choose one of the following: minutes or hours; cm or m; km/h or m/s; etc.
- Set up an equation with the information gained so as to solve for x (or in some instances y as well).
- Basic general knowledge:
  Profit = Selling price – cost price and
  Total cost = Cost per item × number items and
  $\text{speed} = \dfrac{\text{distance}}{\text{time}}$ or $\text{distance} = \text{speed} \times \text{time}$ or $\text{time} = \dfrac{\text{distance}}{\text{speed}}$
  Area and perimeter formulae for circles, triangles and rectangles.
- Conclude by stating your answer clearly. Don't just leave your answer as $x = ...$

EXAMPLE 14 **(Numbers)**

The product of two numbers is 135 and their difference is 6. Determine the two numbers.

Solution

Method 1

Let the larger number be: x
The smaller number will be: $x - 6$ (their difference is 6)
Now use the product statement to set up an equation and solve:
$x \times (x - 6)$ is 135

$\therefore x(x - 6) = 135$

$\therefore x^2 - 6x - 135 = 0$

$\therefore (x - 15)(x + 9) = 0$

$\therefore x = 15$ or $x = -9$

The two numbers are: 15 and 9 or -9 and -15

Method 2

Let the larger number be: x Let the smaller number be: y

Product is 135: $xy = 135$ (A)

Difference is 6: $x - y = 6$ (B)

$\therefore x - 6 = y$ (C)

Substitute C into A

$\therefore x(x-6) = 135$

$\therefore x^2 - 6x - 135 = 0$

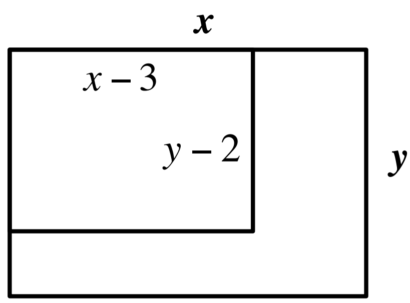
$\therefore (x-15)(x+9) = 0$

$\therefore x = 15$ or $x = -9$

$\therefore y = 15 - 6 = 9$ or $y = -9 - 6 = -15$

The two numbers are: 15 and 9 or -9 and -15

<u>EXAMPLE 15</u> **(Area and perimeter)**



The dimensions of a carpet with a length of x metres and a width of y metres are to be reduced. The length is to be reduced by 3 metres and the width by 2 metres. The area of the new carpet is to be 18m^2. If the perimeter of the original carpet is 48 metres, determine the dimensions of the original carpet.

<u>Solution</u>

The length of the old carpet: x The length of the new carpet: $x - 3$

The width of the old carpet: y The width of the new carpet: $y - 2$

$\therefore 2x + 2y = 48$...A and $(x-3)(y-2) = 18$...B

$\therefore x + y = 24$

$\therefore y = 24 - x$...C

Substitute C in B:

$\therefore (x-3)(24 - x - 2) = 18$

$\therefore (x-3)(22 - x) - 18 = 0$

$\therefore 22x - x^2 - 66 + 3x - 18 = 0$

$\therefore 0 = x^2 - 25x + 84$

$\therefore 0 = (x-21)(x-4)$ $\therefore y = 24 - 21 = 3$ or $y = 24 - 4 = 20$

$\therefore x = 21$ or $x = 4$ The dimensions are 21m by 3m or 4m by 20m

<u>EXAMPLE 16</u> **(Speed distance and time)**

If the speed of Miguel's new car is 10km/h faster than that of his older car, the journey between Rooiplaas and Vaalplaas (100km apart) would be shortened by 30 minutes. Determine the speed of the newer model.

New: R____**V**

100km apart

New car's speed in km/h: x

New car's distance travelled: 100km
New car's time taken in hours: y

Old: R____V

100km apart

Old car's speed in km/h: $x-10$
Old car's distance travelled: 100km
Old car's time taken in hours: $y+\frac{30}{60}$ hours $= y+\frac{1}{2}$

Now express the information for each car as **equations** involving the speed, distance and time formulae**:**

Distance = speed $\times$ time

NEW OLD

$\therefore 100 = xy$...A and $100 = (x-10)(y+\frac{1}{2})$...B

$\therefore y = \frac{100}{x}$...C

Substitute C into B:

$100 = (x-10)(\frac{100}{x}+\frac{1}{2})$

$\therefore 100 = 100 + \frac{1}{2}x - \frac{1000}{x} - 5$ LCD: $2x$

$\therefore 200x = 200x + x^2 - 2000 - 10x$

$\therefore 0 = x^2 - 10x - 2000$

$\therefore 0 = (x-50)(x+40)$

$\therefore x = 50$ or $x = -40$

$\therefore$ Speed of new car: $50 km/h$

EXERCISE 10

1. The product of two consecutive integers is 342. Find the integers.
2. The area of a classroom is $20m^2$. If the length is increased by 3m and the width is increased by 1m, the classroom will double in area. Determine the new dimensions of the new classroom.
3. The area of a rectangle is $96m^2$ and has a perimeter of 39,2m. Determine the dimensions of the rectangle.
4. A lady travels 180km from her wine farm to Ceres. On the return journey she travels at night and travels 30km/h slower. She lost 1 hour on the return journey by travelling slower. At what speed did she travel back home?
5. Mr B is now twice as old as his younger brother. Eight years ago he was three times as old as his younger brother was then. How old is the younger brother now?

REVISION OF LINEAR INEQUALITIES

The following rules are always applicable when working with inequalities:

- Change the direction of the inequality sign whenever you multiply or divide by a negative number.
- Do not change the direction of the inequality if you multiply or divide by a positive number.

EXAMPLE 17

Solve the following inequalities:

(a) $4(x-3) \geq 2(x-10)$

$\therefore 4x-12 \geq 2x-20$

$\therefore 4x-2x \geq -20+12$

$\therefore 2x \geq -8$

$\therefore x \geq -4$

(b) $-4(x-3) \geq 2(x-12)$

$\therefore -4x+12 \geq 2x-24$

$\therefore -4x-2x \geq -24-12$

$\therefore -6x \geq -36$

$\therefore x \leq 6$

EXERCISE 11

Solve the following inequalities:

(a) $x+15 \leq 6-2x$ (b) $x-5 < 2x+3$

(c) $4(x-1) > 6(x-1)$ (d) $4(x-3)-2(x-1) \geq 0$

(e) $\dfrac{y+5}{3} + y \leq 1$ (f) $\dfrac{3y+2}{4} - \dfrac{y-6}{3} > 0$

QUADRATIC INEQUALITIES

A **quadratic inequality** involves determining the values of x for which the graph of a parabola lies either above or below the x-axis. In Chapter 5 you will learn how to sketch the graphs of parabolas in more detail. All we need now is a basic understanding of how to determine the x-intercepts of a parabola as well as its shape.

EXAMPLE 18

Solve for x: $x^2 \leq 4$

Solution

Method 1 **(Graphical approach)**

$x^2 \leq 4$

$\therefore x^2-4 \leq 0$ (First bring all terms to the left)

$\therefore (x+2)(x-2) \leq 0$ (Factorise)

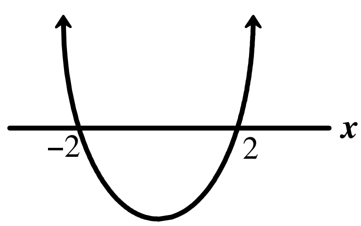
The graph of the parabola $y = x^2 - 4$ is concave (the coefficient of the term in x^2 is positive).

The x-intercepts can be determined as follows:

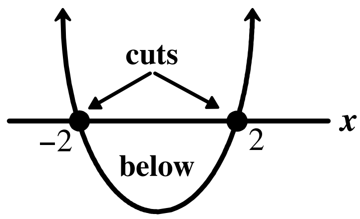
$(x+2)(x-2) = 0$

$\therefore x = -2$ or $x = 2$

Draw a graph of this parabola (just focus on the x-intercepts)



Now determine the values of x for which $x^2 - 4 \leq 0$
This is where the graph **cuts** the x-axis
and where it lies **below** the x-axis.
The solution is clearly between
the two x-intercepts: $-2 \leq x \leq 2$

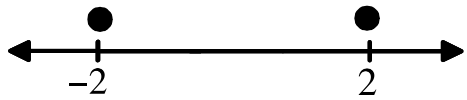


<u>Method 2</u> **(Number line approach)**

$x^2 - 4 \leq 0$ (First bring all terms to the left)

$\therefore (x-2)(x+2) \leq 0$ (Factorise the LHS)

Determine the critical values (x-intercepts) of the quadratic expression on the left.
The critical values are: 2 and -2
The inequality reads: **"smaller than or <u>equal</u> to zero"**
The critical values are included in the final solution. Therefore we will use **closed dots**.



The next thing to do is to determine the values of x for which the expression on the left is **smaller than zero (negative or** < 0 **)**
To do this one has to test values to the left and right of the critical values.

(a) to the left of -2 (values smaller than -2),
(b) between -2 and 2, and
(c) to the right of 2 (values greater than 2)

(a) Values smaller than -2

Try $x = -3$ $(-3-2)(-3+2) = 5$ which is positive. $(+)$

Try $x = -4$ $(-4-2)(-4+2) = 12$ which is positive. $(+)$

You may substitute any value less than -2 and the answer always results in a positive number.

(b) Values between -2 and 2

Try $x = -1$ $(-1-2)(-1+2) = -3$ which is negative. $(-)$

Try $x = 0$ $(0-2)(0+2) = -4$ which is negative. $(-)$

You may substitute any value between -2 and 2 and the answer always results in a negative number.

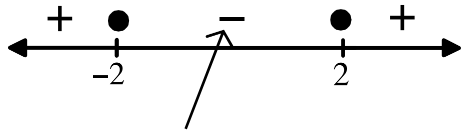
(c) Values greater than 2

Try $x = 3$ $(3-2)(3+2) = 5$ which is positive. $(+)$

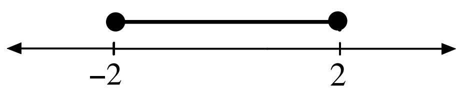
Try $x = 4$ $(4-2)(4+2) = 12$ which is positive. $(+)$

You may substitute any value greater than 2, and you will see that the answer always results in a positive number.

Let us summarise the results of (a), (b) and (c) on the number line.



Therefore the final solution for $x^2 - 4 \leq 0$ is: $-2 \leq x \leq 2$



You may write this in interval notation: $[-2\,;2]$

Method 3 **(Table method)**

$x^2 - 4 \le 0$ (First bring all terms to the left)

$\therefore (x-2)(x+2) \le 0$ (Factorise the LHS)

Determine the critical values (x-intercepts) of the quadratic expression on the left.
The critical values are: 2 and -2 (This is where $x-2=0$ and $x+2=0$)
Now draw a table with the critical values and the factors as shown in the table.
The value of the expression $x-2$ is negative ($-$) for all $x<2$ and positive ($+$) for all $x>2$
The value of the expression $x+2$ is negative ($-$) for all $x<-2$ and positive ($+$) for all $x>-2$

		-2		2	
$x-2$	$-$		$-$	0	$+$
$x+2$	$-$	0	$+$		$+$
$(x-2)(x+2)$	$+$	0	$-$	0	$+$

The inequality $(x-2)(x+2) \le 0$ reads: **"smaller than or equal to zero"**
The product $(x-2)(x+2)$ is negative (smaller than zero) or equal to zero in the interval $-2 \le x \le 2$
Therefore the final solution for $x^2 - 4 \le 0$ is: $-2 \le x \le 2$

EXAMPLE 19

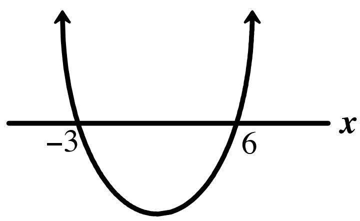
Solve for x in the inequality $x^2 - 3x > 18$

Solution

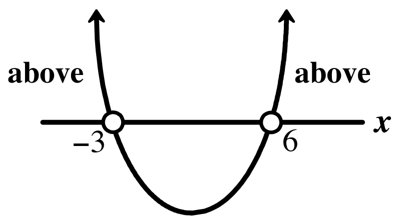
Method 1 **(Graphical approach)**

$x^2 - 3x - 18 > 0$ (First bring all terms to the left)

$\therefore (x-6)(x+3) > 0$ (Factorise)



The graph of the parabola $y = x^2 - 3x - 18$ is concave (the coefficient of the term in x^2 is positive).
The x-intercepts can be determined as follows:
$(x-6)(x+3) = 0$
$\therefore x = 6$ or $x = -3$
Draw a graph of this parabola (just focus on the x-intercepts)
Now determine the values of x for which $x^2 - 3x - 18 > 0$
This is where the graph lies **above** the x-axis but doesn't cut the x-axis.



The solution is clearly to the left of -3 and to the right of 6, i.e. $x < -3$ or $x > 6$

Method 2 **(Number line approach)**

$x^2 - 3x - 18 > 0$ (First bring all terms to the left)

$\therefore (x-6)(x+3) > 0$ (Factorise the LHS)

Determine the critical values (x-intercepts) of the quadratic expression on the left.
The critical values are: 6 and -3
The inequality reads: **"greater than zero (not equal to)"**
The critical values are not included in the final solution. Therefore, we will use **open dots**.

The next thing to do is to determine the values of x for which the expression on the left is **smaller than zero (negative or <0)**.
To do this one has to test values to the left and right of the critical values.

(a) to the left of -3 (values smaller than -3),
(b) between -3 and 6, and
(c) to the right of 6 (values greater than 6)

(a) Values smaller than -3

Try $x=-4$ $\quad (-4-6)(-4+3)=10$ which is positive. $(+)$

Try $x=-5$ $\quad (-5-6)(-5+3)=22$ which is positive. $(+)$

You may substitute any value less than -3 and the answer always results in a positive number.

(b) Values between -3 and 6

Try $x=-2$ $\quad (-2-6)(-2+3)=-8$ which is negative. $(-)$

Try $x=0$ $\quad (0-6)(0+3)=-18$ which is negative. $(-)$

Try $x=4$ $\quad (4-6)(4+3)=-14$ which is negative. $(-)$

You may substitute any value between -3 and 6 and the answer always results in a negative number.

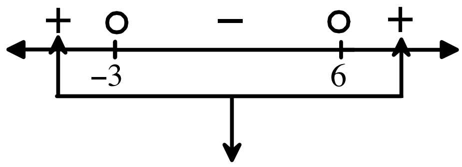
(c) Values greater than 6

Try $x=7$ $\quad (7-6)(7+3)=10$ which is positive. $(+)$

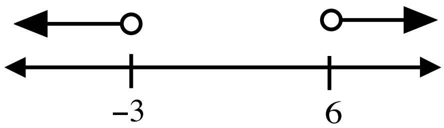
Try $x=8$ $\quad (8-6)(8+3)=22$ which is positive. $(+)$

You may substitute any value greater than 6, and you will see that the answer always results in a positive number.

Let us summarise the results of (a), (b) and (c) on the number line.



Therefore the final solution for $x^2-3x-18>0$ is: $\quad x<-3$ or $x>6$



You may write this in interval notation: $x\in(-\infty;-3)\cup(6;\infty)$

<u>Method 3</u> **(Table method)**

$x^2-3x-18>0$ (First bring all terms to the left)

$\therefore (x-6)(x+3)>0$ (Factorise the LHS)

Determine the critical values (x-intercepts) of the quadratic expression on the left.
The critical values are: 6 and -3 (This is where $x-6=0$ and $x+3=0$)
Now draw a table with the critical values and the factors as shown in the table.

The value of the expression $x+3$ is negative ($-$) for all $x<-3$ and positive ($+$) for all $x>-3$
The value of the expression $x-6$ is negative ($-$) for all $x<6$ and positive ($+$) for all $x>6$

		-3		6	
$x-6$	$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$
$(x-6)(x+3)$	$+$	0	$-$	0	$+$

The inequality $(x-6)(x+3)>0$ reads: **"greater than zero (not equal to)"**
The product $(x-6)(x+3)$ is positive in the interval $x<-3$ or $x>6$
Therefore the final solution for $(x-6)(x+3)>0$ is: $x<-3$ or $x>6$

<u>EXAMPLE 20</u>

Solve for x in the inequality $x-2x^2 \geq 0$

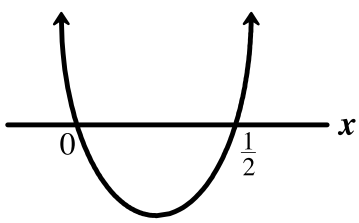
<u>Solution</u>

<u>Method 1</u> **(Graphical approach)**

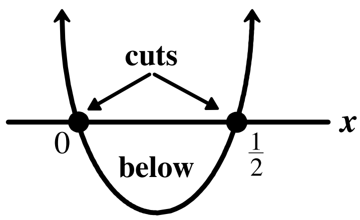
$x-2x^2 \geq 0$
$\therefore -2x^2+x \geq 0$
$\therefore 2x^2-x \leq 0$ (Make the coefficient of x^2 positive by multiplying by -1)
$\therefore x(2x-1) \leq 0$ (Factorise)

The graph of the parabola $y=2x^2-x$ is concave (the coefficient of the term in x^2 is positive).
The x-intercepts can be determined as follows:
$x(2x-1)=0$

$\therefore x=0$ or $x=\frac{1}{2}$



Draw a graph of this parabola (just focus on the x-intercepts)
Now determine the values of x for which $x(2x-1) \leq 0$.
This is where the graph **cuts** the x-axis and where it lies **below** the x-axis.
The solution is clearly between the two x-intercepts: $0 \leq x \leq \frac{1}{2}$

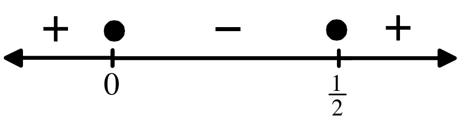


<u>Method 2</u> **(Number line approach)**

$x-2x^2 \geq 0$
$\therefore -2x^2+x \geq 0$
$\therefore 2x^2-x \leq 0$ (Make the coefficient of x^2 positive by multiplying by -1)
$\therefore x(2x-1) \leq 0$ (Factorise)

Draw a number line and test values as done previously.

The solution is: $0 \leq x \leq \frac{1}{2}$

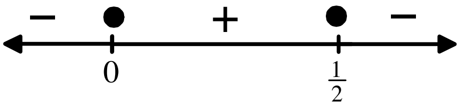


Alternatively:

$x - 2x^2 \geq 0$

$\therefore x(1-2x) \geq 0$ (Factorise without multiplying by -1)

If you now test values as done previously, the pattern of signs will be as follows:



Therefore the final solution for $x - 2x^2 \geq 0$ is: $0 \leq x \leq \frac{1}{2}$

<u>**Method 3**</u> **(Table method)**

$x - 2x^2 \geq 0$

$\therefore -2x^2 + x \geq 0$

$\therefore 2x^2 - x \leq 0$ (Make the coefficient of x^2 positive by multiplying by -1)

$\therefore x(2x-1) \leq 0$ (Factorise)

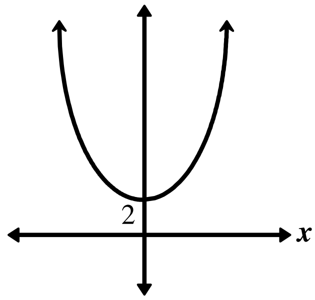
Now draw a table with the critical values and the factors as shown in the table.

The solution is: $0 \leq x \leq \frac{1}{2}$

		0		$\frac{1}{2}$	
x	−	0	+		+
$2x-1$	−		−	0	+
$x(2x-1)$	+	0	−	0	+

<u>**Special cases**</u>

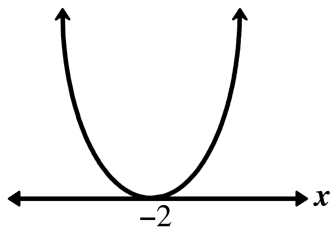
(a) Solve for x: $x^2 + 2 > 0$



The parabola doesn't cut the x-axis. It lies above the x-axis for all real values of x.

The solution is: $x \in \{\text{real numbers}\}$

(b) Solve for x: $(x+2)^2 \geq 0$



The parabola cuts the x-axis at -2 and also lies above the x-axis for all other real values of x

The solution is: $x \in \{\text{real numbers}\}$

(c) Solve for x: $(x+2)^2 > 0$

Since we are only concerned with the values of x for which the parabola is above the x-axis, the solution is: $x < -2$ or $x > -2$

(d) Solve for x: $(x+2)^2 \leq 0$

The parabola doesn't lie below the x-axis but cuts it at $x = -2$

The solution is: $x = -2$

(e) Solve for x: $(x+2)^2 < 0$

Here we want the values of x for which the parabola lies below the x-axis excluding where it cuts the x-axis.

Clearly there is no solution to this inequality.

EXERCISE 12

1. Solve the following inequalities:

(a)	$x^2 \leq 16$	(b)	$x^2 \geq 25$	(c)	$x^2 < 9$
(d)	$x^2 > 1$	(e)	$x^2 - x \geq 0$	(f)	$x^2 - x < 12$
(g)	$x^2 - 5x \leq 6$	(h)	$4x^2 > 9$	(i)	$2x^2 - 5x - 3 < 0$
(j)	$(1-2x)(x+3) \leq 0$	(k)	$(x-3)(x-4) \geq 12$	(l)	$2x + 3 \geq x^2$
(m)	$3x + 9 > 2x^2$	(n)	$x^2 + 4 > 0$	(o)	$x^2 + 4 < 0$
(p)	$(x+4)^2 > 0$	(q)	$(x+4)^2 \geq 0$	(r)	$(x+4)^2 \leq 0$
(s)	$(x+4)^2 < 0$	(t)	$9 \geq -x(x-6)$		
(u)	$x^2 \leq 5$ (surd form)	(v)	$3x - x^2 \geq 0$ where $x \in \mathrm{N}$		

2. Determine for what values of x each of the following expressions will be non-real.
(Hint: Non real number = $\sqrt{\text{negative number}}$. Therefore if:
expression = $\sqrt{(\text{inside expression})}$ then it would be non-real for inside expression < 0)

(a) $\sqrt{x}$ (b) $\sqrt{x-3}$ (c) $\sqrt{x^2 + x}$

3. For what values of x will $\sqrt{x^2 - 25}$ be real.

4. (a) Show by completing the square that $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

(b) Hence solve for x if:
(1) $x^2 + x + 1 > 0$ (2) $x^2 + x + 1 < 0$

REVISION EXERCISE

1. Solve for x:
(Leave answers correct to two decimal places where appropriate)

(a)	$x^2 - 7x - 8 = 0$	(b)	$x^2 - 7x + 8 = 0$
(c)	$9x^2 = 4x$	(d)	$\frac{15}{x-1} - \frac{x+3}{x^2+x} = \frac{12}{x}$
(e)	$x^2 \geq 64$	(f)	$x^2 + 1 \geq 0$

(g) $\frac{4}{x-6} = \frac{2-3x}{2x-x^2} - \frac{12}{x^2-8x+12}$ (correct to one decimal place)

(h) $16x^2 + 1 = 0$ (i) $0 < 2x^2 - x$

2. Show by completing the square that $x^2 + 5x - 10 = \left(x + \frac{5}{2}\right)^2 - \frac{65}{4}$

3. Show that $-x^2 - 10x + 12$ has a maximum value of 37.

4. Discuss the nature of the roots of $-x^2 + 2x = -2$

5. Determine the values of k for which the equation $\frac{1}{k} = x^2 - x + 1$ where $k \neq 0$ has real roots.

6. (a) Solve: $p - \frac{2}{p+2} = 13$

(b) Hence solve for x if: $x^2 - 4x - 1 - \frac{2}{x^2 - 4x + 1} = 13$

7. (a) Show that $\text{P} = 3x^2 - 6x + 12$ is positive for all values of x.

(b) Hence, or otherwise, state for which values of x will $\sqrt{3x^2 - 6x + 12}$ be real.

8. For which values of x will $\text{Q} = \sqrt{x^2 - 5x + 6}$ be non-real?

9. Solve for x and y: $x^2 + 2yx = 3y^2$ and $2y - x = 6$

10. Solve for x if $\sqrt{x-2} = x - 4$. Verify your results.

11. Consider the inequality: $\frac{x-1}{x+3} \leq 0$

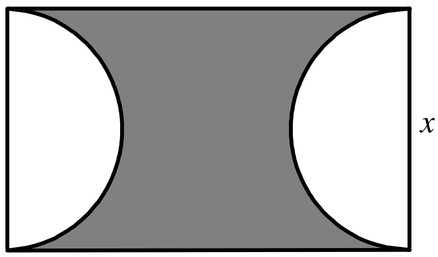
(a) Why is it a problem to multiply both sides of the inequality by the LCD $= x + 3$?

(b) Why is it acceptable to multiply both sides by $(x+3)^2$?

(c) By multiplying both sides by $(x+3)^2$, solve the inequality.

SOME CHALLENGES

1. A new design for a rectangular table place has been implemented by a company. Refer to the diagram below for more detail. The shaded area below is shaded with a multi-coloured pattern. The length of the place mat remains fixed at 100mm, while the width of the mat can vary in length.



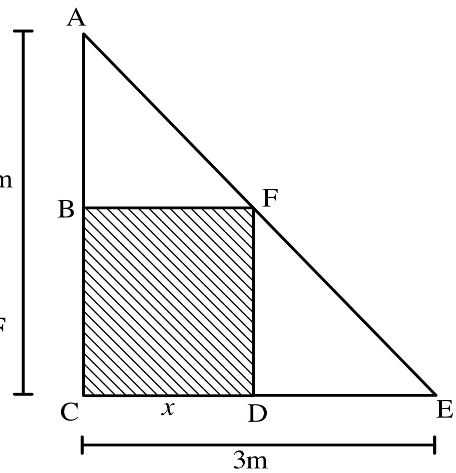
(a) Show that the shaded area is given by $\text{A}(x) = 100x - \frac{\pi}{4}x^2$

(b) Determine the maximum shaded area possible.

2. Calculate x if $x = \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 \ldots\ldots}}}}}$

3. Calculate x if $2 = x^{x^{x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}}}}$ and $x > 0$

4. In the diagram below, BCDF is an inscribed rectangle in triangle ACE.
AC = 4m and CE = 3m
(Hint: Draw axes on AC and CE and let C be the point $(0;0)$)



(a) Determine the area of the rectangle BCDF if BC = CD.

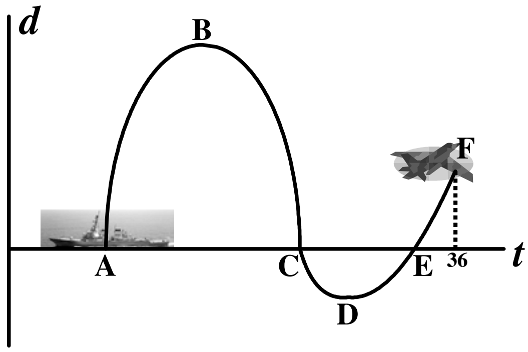
(b) Determine the maximum area of rectangle BCDF.

5. A missile is launched from a military ship at A on the ocean. The missile reaches B above the water and then drops downwards and enters the sea at C. It then moves downwards to D below sea level and then turns and moves upwards and leaves the sea at E. It then hits its target at F.

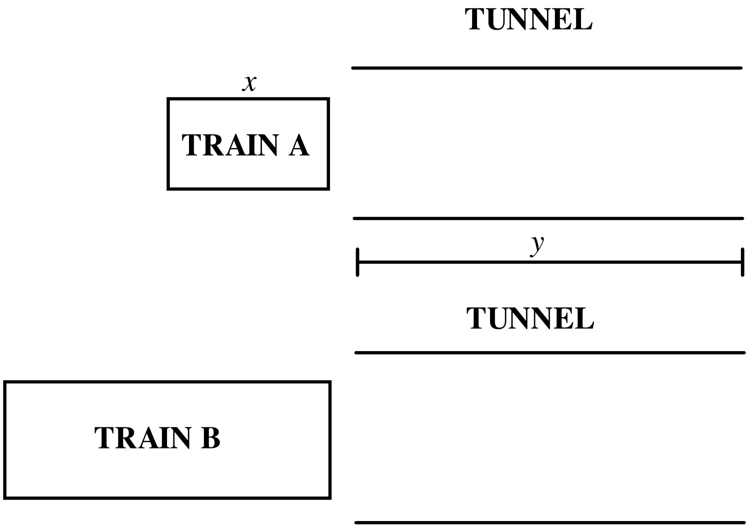
The equation of the movement of the missile from A to C is defined by $d = -2t^2 + 32t - 96$. The equation of the movement of the missile from C to F is defined by $d = \frac{1}{36}t^2 - t + 8$ where t represents the time moved by the missile in seconds and d represents the displacement of the missile from sea level in kilometres. The target is hit after 36 seconds.

(a) Calculate the times that the missile was above sea level.

(b) What was the displacement of the missile above sea level when it hit its target?



6. TRAIN A **passes completely** through a tunnel in 4 minutes. A second train, TRAIN B, twice as long as the first, **passes completely** through the tunnel in 5 minutes. Both trains are travelling at the same speed, namely, 48km/h. Consider the diagram below.



In the diagram y represents the length of the tunnel and x the length of TRAIN A.

[Additional information: $\text{TIME} = \frac{\text{DISTANCE}}{\text{SPEED}}$]

(a) Determine the length of TRAIN A in km

(b) Determine the length of the tunnel in km

CHAPTER 3 – NUMBER PATTERNS

REVISION OF LINEAR NUMBER PATTERNS

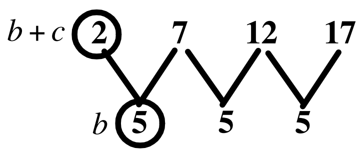
In Grade 10 we dealt with linear number patterns having a general term of the form $T_n = bn + c$.

EXAMPLE

Consider the following linear number pattern:
2 ; 7 ; 12 ; 17 ; ………………….
(a) Determine the nth term and hence the 199th term.
(b) Which term of the number pattern is equal to 497?

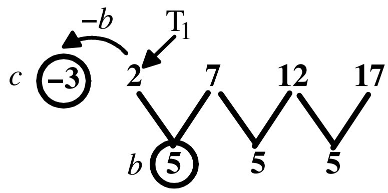
Solutions

(a) **Method 1**



$b = 5$ $\qquad b + c = 2$
$\therefore 5 + c = 2$
$\therefore c = -3$
$T_n = bn + c$
$\therefore T_n = 5n - 3$
$\therefore T_{199} = 5(199) - 3 = 992$

Method 2



$b = 5$
$c = T_1 - b = 2 - 5 = -3$
$T_n = bn + c$
$\therefore T_n = 5n - 3$
$\therefore T_{199} = 5(199) - 3 = 992$

Method 3

Multiples of $b = 5$	5	10	15	20
What to do to get original number	−3	−3	−3	−3
Original pattern	2	7	12	17

The constant number being subtracted in the second row represents the value of c.
$\therefore T_n = 5n - 3$ $\qquad [b = 5 \text{ and } c = -3]$
$\therefore T_{199} = 5(199) - 3 = 992$

EXERCISE 1

1. Consider the following linear number pattern:
   $3 ; 4 ; 5 ; 6 ;$ ....
   (a) Determine the nth term and hence the 100th term.
   (b) Which term of the number pattern is equal to 52?

2. Consider the following linear number pattern:
$3 ; 5 ; 7 ; 9 ; ....$
(a) Determine the nth term and hence the 180th term.
(b) Which term of the number pattern is equal to 241?

3. Consider the following linear number pattern:
$-2 ; 3 ; 8 ; 13 ; ....$
(a) Determine the nth term and hence the 259th term.
(b) Which term of the number pattern is equal to 753?

4. Consider the following linear number pattern:
$4 ; 2 ; 0 ; -2 ; ....$
(a) Determine the nth term and hence the 150th term.
(b) Which term of the number pattern is equal to -794?

5. Consider the following linear number pattern:
$-6 ; -10 ; -14 ; -18 ; ....$
(a) Determine the nth term and hence the 600th term.
(b) Which term of the number pattern is equal to -442?

6. Consider the following linear number pattern:
$1\frac{1}{2} ; \ 2 \ ; \ 2\frac{1}{2} \ ; \ 3 \ ; ....$
(a) Determine the nth term and hence the 130th term.
(b) Which term of the number pattern is equal to 41?

QUADRATIC NUMBER PATTERNS

We will now focus on **quadratic number patterns** with general terms of the form $T_n = an^2 + bn + c$.

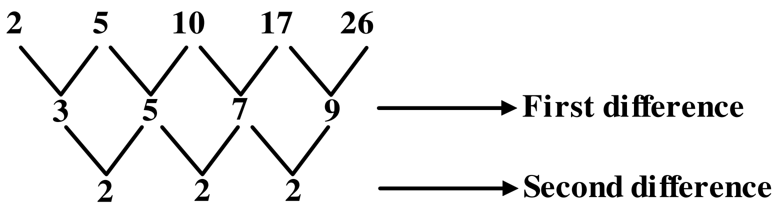
EXAMPLE 1

Consider, for example, the formula $T_n = n^2 + 1$.
We can generate the terms of the number sequence as follows:

$T_1 = (1)^2 + 1 = 2 \qquad T_2 = (2)^2 + 1 = 5 \qquad T_3 = (3)^2 + 1 = 10$

$T_4 = (4)^2 + 1 = 17 \qquad T_5 = (5)^2 + 1 = 26$

If we now consider the pattern 2; 5; 10; 17; 26; ………, we can investigate some interesting properties of this quadratic number pattern.
If we get the differences between the terms, it is clear that this number pattern does not have a constant first difference as with linear number patterns. However, it does have a constant second difference. Number patterns with constant second differences are called **quadratic number patterns**.



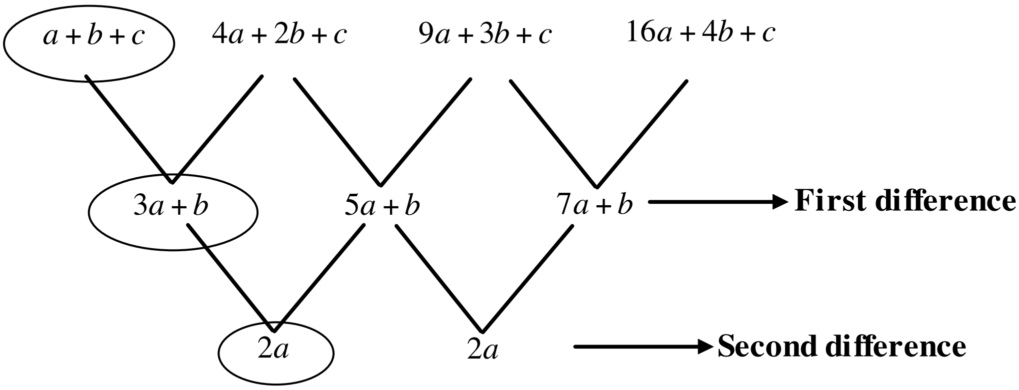
The question now arises as to how we would be able to determine the general term of any given quadratic number pattern.

Suppose that the general term of a particular quadratic number pattern is given by $T_n = an^2 + bn + c$.

The terms of the number pattern would then be:

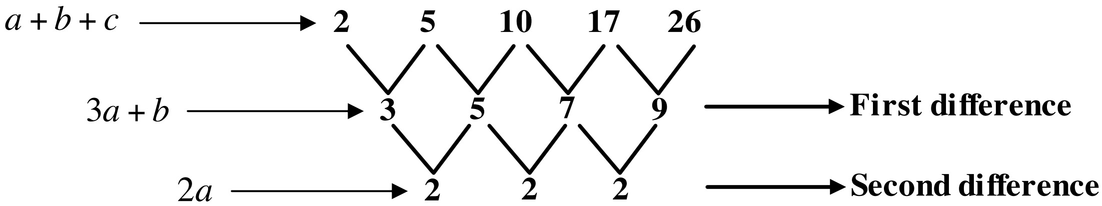
$T_1 = a(1)^2 + b(1) + c = a + b + c$ $\qquad$ $T_2 = a(2)^2 + b(2) + c = 4a + 2b + c$

$T_3 = a(3)^2 + b(3) + c = 9a + 3b + c$ $\qquad$ $T_4 = a(4)^2 + b(4) + c = 16a + 4b + c$



You will notice that the constant second difference is given by the expression $2a$. The first term in the first difference row is given by $3a + b$ and the first term is given by $a + b + c$.

So, consider the previous number pattern: 2; 5; 10; 17; 26; ……..



It is clearly a quadratic number pattern because it has a constant second difference. You can now proceed as follows:

$2a = 2$	$3a + b = 3$	$a + b + c = 2$
$\therefore a = 1$	$\therefore 3(1) + b = 3$	$\therefore 1 + 0 + c = 2$
	$\therefore b = 0$	$\therefore c = 1$

Therefore the general term is $T_n = n^2 + 0n + 1 = n^2 + 1$

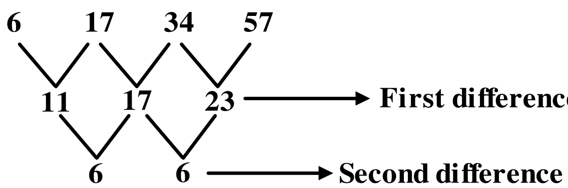
<u>EXAMPLE 2</u>

Consider the following number pattern: 6; 17; 34; 57; ……………………

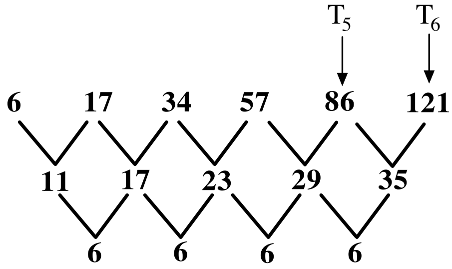
(a) Show that it is a quadratic number pattern.
(b) Write down the next two terms of the number pattern.
(c) Hence determine the *n*th term as well as the 100th term.
(d) Determine which term equals 162.

<u>Solution</u>

(a)



(b) The fourth term in the second row can be determined by adding 6 to 23 to get 29. It will then be possible to determine the fifth term of the number pattern by adding 29 to 57 which gives 86. This process is repeated to give the sixth term of the number pattern.



(c) $2a$ = second difference

$\therefore 2a = 6$ $\quad 3a + b = 11$ $\quad a + b + c = 6$

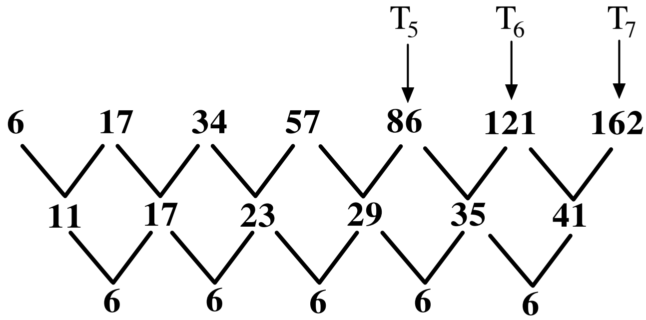
$\therefore a = 3$ $\quad \therefore 3(3) + b = 11$ $\quad \therefore 3 + 2 + c = 6$

$\therefore b = 2$ $\quad \therefore c = 1$

$\therefore T_n = 3n^2 + 2n + 1$

$\therefore T_{100} = 3(100)^2 + 2(100) + 1 = 30201$

(d) You could have continued the process done in (b) to find out which term equals 162. Clearly, the 7th term is equal to 162.



Alternatively, you could have used quadratic equations to assist you:

$T_n = 162$

$\therefore 3n^2 + 2n + 1 = 162$

$\therefore 3n^2 + 2n - 161 = 0$

$\therefore (3n + 23)(n - 7) = 0$

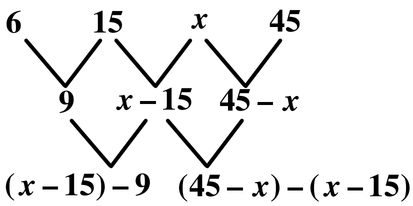
$\therefore n = -\frac{23}{3}$ or $n = 7$

But $n \neq -\frac{23}{3}$

$\therefore n = 7$

EXAMPLE 3

$6 ; 15 ; x ; 45 ; ....$ is a quadratic number pattern (sequence). Determine the value of x.



$(x - 15) - 9 = (45 - x) - (x - 15)$

$\therefore x - 24 = 45 - x - x - 15$

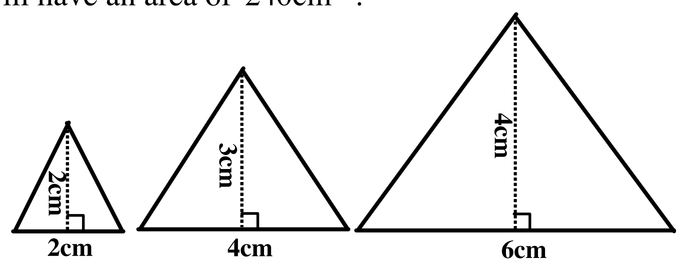
$\therefore x - 24 = 60 - 2x$

$\therefore 3x = 84$

$\therefore x = 28$

EXERCISE 2

1. Consider the following number pattern:
   $6\,;13\,;22\,;33\,;$ ....
   (a) Show that it is a quadratic number pattern.
   (b) Write down the next two terms of the number pattern.
   (c) Hence determine the *n*th term as well as the 100th term.
   (d) Determine which term equals 118.
2. Consider the following number pattern:
   $2\,;13\,;32\,;59\,;$ ....
   (a) Show that it is a quadratic number pattern.
   (b) Write down the next two terms of the number pattern.
   (c) Hence determine the *n*th term as well as the 160th term.
   (d) Determine which term equals 389.
3. Consider the following number pattern:
   $-1\,;-10\,;-25\,;-46\,;$ ....
   (a) Show that it is a quadratic number pattern.
   (b) Write down the next two terms of the number pattern.
   (c) Hence determine the *n*th term as well as the 80th term.
   (d) Determine which term equals -7498.
4. For each of the following number patterns determine the general term (or the *n*th term) and hence the 150th term.
   (a) $4\,;7\,;12\,;19\,;$ .... (b) $0\,;3\,;8\,;15\,;$ ....
   (c) $2\,;-4\,;-14\,;-28\,;$ .... (d) $6\,;10\,;14\,;18\,;$.............
   (e) $3\,;12\,;27\,;48\,;$ .... (f) $7\,;24\,;51\,;88\,;$ ....
   (g) $3\frac{1}{2}\,;10\frac{1}{2}\,;21\frac{1}{2}\,;36\frac{1}{2}\,;$ .... *(h) $\frac{2}{3}\,;\frac{5}{6}\,;\frac{10}{11}\,;\frac{17}{18}\,;$........
5. Consider the quadratic number pattern: $1\,;7\,;x\,;31\,;$ ....
   (a) Determine the value of x.
   (b) Determine the *n*th term of the sequence.
6. Consider the quadratic number pattern: $-4\,;x\,;-28\,;-46\,;$ ....
   (a) Determine the value of x.
   (b) Determine the *n*th term of the sequence.
7. The constant second difference of the quadratic number pattern:
   $4\,;x\,;8\,;y\,;20\,;$ .... is 2.
   (a) Determine the value of x and y.
   (b) Determine which term equals 125.
8. A sequence of isosceles triangles is drawn. The first triangle has a base of 2cm and height of 2cm. The second triangle has a base that is 2cm longer than the base of the first triangle. The height of the second triangle is 1cm longer than the height of the first triangle. This pattern of enlargement will continue with each triangle that follows.
   (a) Determine the area of the 100^{th} triangle.
   (b) Which triangle will have an area of 240cm^2 ?



REVISION EXERCISE

1. Determine the nth term and hence the 16th term for each of the following number patterns:
   (a) $5;8;11;14;$.......... (b) $3;9;17;27;$............
   (c) $2;6;10;14;$......... (d) $2;6;12;20;$.............

2. The following number pattern is quadratic:
   $-4;-3;-4;-7;-12;$................
   (a) Determine the general term for the first difference row.
   (b) Determine the first difference between the 25th and 26th terms of the quadratic sequence.
   (c) Determine the general term for the quadratic pattern.
   (d) Which term of the quadratic pattern is equal to -67?

3. In a quadratic number pattern, the second term is 1, the third term is -6 and the fifth term is -14. Calculate the first term and the constant second difference.

4. Consider the following number pattern:
   $1=1$

   $1+2=3$

   $1+2+3=6$

   $1+2+3+4=10$

   $1+2+3+4+5=15$
   (a) Determine a general rule in terms of n for evaluating:
   $1+2+3+4+5+6+\ldots\ldots\ldots\ldots+n$
   (b) Hence calculate the value of:
   $1+2+3+4+5+6+7+\ldots\ldots\ldots\ldots+200$

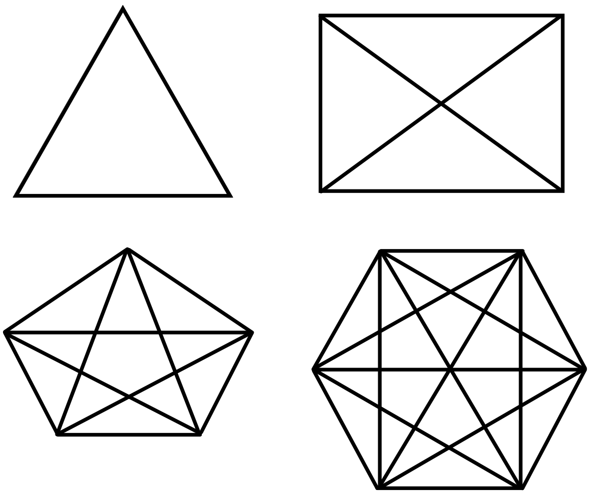
5. A biologist investigating the growth rate of a new insect species recorded, on the hour, the total number of insects present. She recorded this information in the following table. At 08h00 on Monday, after one hour had passed, she noticed that there were three insects present. At 09h00, there were seven insects present. Unfortunately she forgot to record the total number of insects at 10h00. What she realised is that the total number of insects present on the hour followed a quadratic number pattern.

Time	Hours passed (n)	Total number of insects ($\mathbf{T}_n$)
08h00	1	3
09h00	2	7
10h00	3	x
11h00	4	21
12h00	5	31
13h00	6	43

   (a) Determine the value of x, the total number of insects present after 3 hours.
   (b) Determine the total number of insects at 21h00 on that Monday evening.

EXAMPLE 4 (ENRICHMENT)

Roxy was given the task of determining the number of diagonals (not forming the sides of the figure) that can be drawn in an n-sided polygon. She recorded her results in the table which follows.

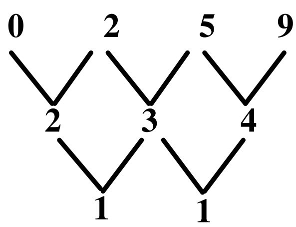


Number of sides	3	4	5	6	7
Number of diagonals drawn from each vertex	0	1	2	3	a
Total number of diagonals	0	2	5	9	b

(a) Write down the value of a and b.
(b) Determine the number of diagonals drawn from each vertex in an n- sided polygon.
(c) Determine the total number of diagonals that can be drawn in an n-sided polygon.
(d) Determine the number of diagonals that can be drawn in a ten-sided polygon.

Solution

(a) $a = 4$ and $b = 14$
(b) There are $n - 3$ diagonals drawn from n sides.
(c)



$2a = 1$ $3a + b = 2$ $\frac{1}{2} + \frac{1}{2} + c = 0$

$\therefore a = \frac{1}{2}$ $\therefore 3(\frac{1}{2}) + b = 2$ $\therefore c = -1$

$\therefore b = \frac{1}{2}$

$\therefore T_n = \frac{1}{2}n^2 + \frac{1}{2}n - 1$

However, the pattern is only applicable from 3 sides onwards. This means that $n = 1$ and $n = 2$ is missing. Therefore we need to subtract 2 from n in the general term.
Therefore the number of diagonals (N) in an n-sided polygon is given by the formula:

$N_n = \frac{1}{2}(n-2)^2 + \frac{1}{2}(n-2) - 1$

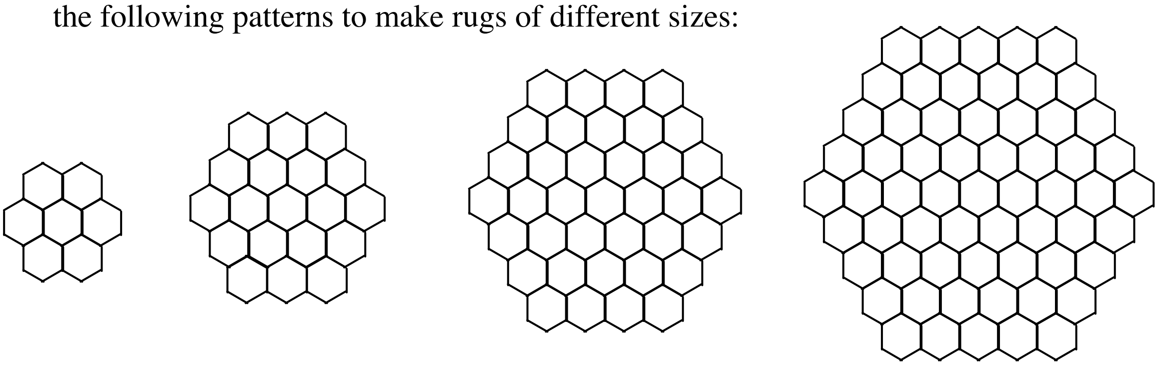
This formula is the general term of this number pattern. This formula will determine the total number of diagonals that can be drawn in an n-sided polygon.

(d) $N_{10} = \frac{1}{2}(10-2)^2 + \frac{1}{2}(10-2) - 1$

$\therefore T_{10} = 35$

SOME CHALLENGES

1. Six-sided carpet tiles are used to make floor rugs. The tiles are arranged in the following patterns to make rugs of different sizes:



Pattern 2 Pattern 3 Pattern 4 Pattern 5

(a) How many tiles will be needed to make pattern 6?
(Pattern 1 is excluded because it only consists of one carpet tile).
(b) Now determine a general formula to determine the number of tiles in any pattern.

2. The distance of an object (in metres) from a starting point at a particular time (in seconds) is recorded in the table below:

Time in seconds	0	1	2	3	4	5	6
Distance from starting point	3	4	9	18			

(a) Complete the table for the next three seconds.
(b) The relationship between the distance travelled (d) in a particular time (t) is modelled by the equation $d = at^2 + bt + c$.
(1) Determine the value of c when $t = 0$.
(2) Determine the value of a and b.
(c) Determine how far from the starting point the object is after 10 seconds.

CHAPTER 4 – ANALYTICAL GEOMETRY

Analytical Geometry deals with the study of geometry using the Cartesian Plane. It is an algebraic approach to the study of geometry. In this chapter we will deal with:

- Revision of Grade 10 concepts (distance between two points; midpoint of a line segment; gradient of a line between two points)
- Inclination of a line
- Equation of a straight line
- Applications involving quadrilaterals

REVISION OF GRADE 10 CONCEPTS

In Grade 10 the formulae for the **distance** between two points, the **midpoint** of a line segment and the **gradient** of a line were established. We will now revise these formulae and their use. Please note that these formulae can also be used to determine the coordinates of points.

DISTANCE BETWEEN TWO POINTS

The distance formula can be used to determine the length of a line segment between two points or the coordinates of a point when the length is known.

The formula to calculate the length of a line segment between two points $\text{A}(x_\text{A}; y_\text{A})$ and $\text{B}(x_\text{B}; y_\text{B})$ is given by the formula:

$$\text{AB}^2 = (x_\text{B} - x_\text{A})^2 + (y_\text{B} - y_\text{A})^2 \text{ or}$$

$$\text{AB} = \sqrt{(x_\text{B} - x_\text{A})^2 + (y_\text{B} - y_\text{A})^2}$$

MIDPOINT OF A LINE SEGMENT

The formula for point M, the midpoint of a line segment AB joining the points $\text{A}(x_\text{A}; y_\text{A})$ and $\text{B}(x_\text{B}; y_\text{B})$ is given by the formula:

$$\text{M}(x_\text{M}; y_\text{M}) = \text{M}\left(\frac{x_\text{B} + x_\text{A}}{2}; \frac{y_\text{B} + y_\text{A}}{2}\right)$$

GRADIENT OF A LINE

The gradient of a line between any two points on the line is the ratio:

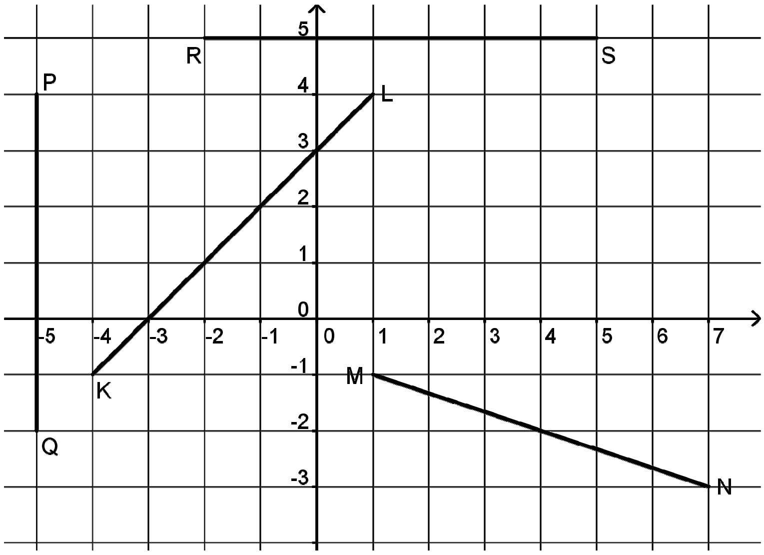
$$m = \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}$$

A formula to calculate the gradient of a line joining two points $\text{A}(x_\text{A}; y_\text{A})$ and $\text{B}(x_\text{B}; y_\text{B})$ is given by the formula:

The gradient of line AB: $m_\text{AB} = \dfrac{y_\text{B} - y_\text{A}}{x_\text{B} - x_\text{A}}$

EXAMPLE 1 **(Basic use of each of the above formulae)**

In the diagram below line segments KL, MN, PQ and RS are sketched.



(a) Calculate the lengths of MN and RS

Solutions

$MN^2 = (x_N - x_M)^2 + (y_N - y_M)^2$ and $RS^2 = (x_S - x_R)^2 + (y_S - y_R)^2$

$\therefore MN^2 = (7-1)^2 + (-3-(-1))^2$ $\quad\therefore RS^2 = (5-(-2))^2 + (5-5)^2$

$\therefore MN^2 = 36+4$ $\quad\therefore RS^2 = 49+0$

$\therefore MN^2 = 40$ $\quad\therefore RS^2 = 49$

$\therefore MN = \sqrt{40}$ $\quad\therefore RS = 7$

Alternatively RS could have been calculated by only considering the distance between the x-values. This is possible because RS is a horizontal line.

$\therefore RS = x_S - x_R = 5-(-2) = 7$

(b) Determine the midpoint of KL and PQ

Solutions

Let T be the midpoint of KL and that V the midpoint of PQ.

$T\left(\dfrac{x_L + x_K}{2}; \dfrac{y_L + y_K}{2}\right)$ and $V\left(\dfrac{x_Q + x_P}{2}; \dfrac{y_Q + y_P}{2}\right)$

$\therefore T\left(\dfrac{1+(-4)}{2}; \dfrac{4+(-1)}{2}\right)$ and $V\left(\dfrac{(-5)+(-5)}{2}; \dfrac{(-2)+4}{2}\right)$

$\therefore T\left(-\dfrac{3}{2}; \dfrac{3}{2}\right)$ and $V(-5;1)$

(c) State which line has a:

(i) negative gradient (ii) positive gradient

(iii) gradient of zero (iv) undefined gradient

Solutions

(i) MN has a negative gradient (sloping down from left to right)

(ii) KL has a positive gradient (sloping up from left to right)

(iii) RS has gradient of zero (horizontal line)

(iv) PQ's gradient is undefined (vertical line)

(d) Determine the gradient of KL and MN

Solutions

$\text{gradient}_{KL} = \dfrac{y_L - y_K}{x_L - x_K}$ and $\text{gradient}_{MN} = \dfrac{y_N - y_M}{x_N - x_M}$

$\therefore \text{gradient}_{KL} = \dfrac{4-(-1)}{1-(-4)} = \dfrac{5}{5} = 1$ and $\text{gradient}_{MN} = \dfrac{-3-(-1)}{7-1} = -\dfrac{2}{6} = -\dfrac{1}{3}$

REVISION EXERCISE (Revision of the basic Grade 10 formulae)

1. Determine the length of the line segment joining each pair of points:
   (a) $A(1;-4)$ and $B(-2;-7)$. (b) $A(3;0)$ and $B(-6;3)$.
   (c) $A(-2;1)$ and $B(3;13)$. (d) $A(5;-3)$ and $B(-1;-3)$.
2. (a) Determine the perimeter of ΔABC with $A(2;3)$, $B(3;-2)$ and $C(-2;-3)$.
   (b) Show that ΔABC is a right-angled triangle (Hint: Use Pythagoras)
3. Calculate the coordinates of the midpoint of the line joining the points.
   (a) $A(1;-4)$ and $B(-2;-7)$ (b) $P(3;0)$ and $Q(-6;3)$.
   (c) $(-2;1)$ and $(3;13)$. (d) $R(2;3)$ and $S(2;-9)$.
4. Calculate the gradients of the lines joining the following pairs of points.
   (a) $(1;-4)$ and $(-2;-7)$. (b) $(3;0)$ and $(-6;3)$.
   (c) $(-5;1)$ and $(-5;6)$. (d) $(-2;1)$ and $(3;1)$.
   (e) $(2p;q)$ and $(p;q-2)$

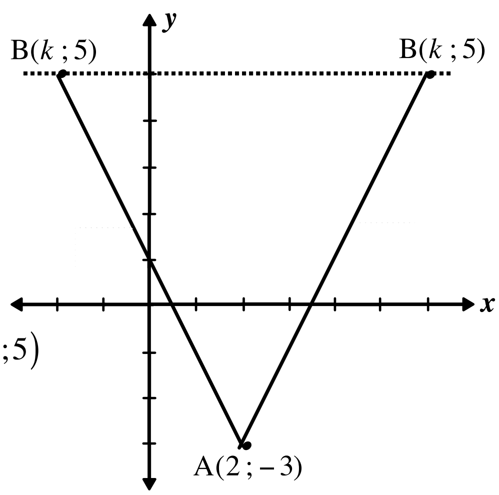
EXAMPLE 2

In this example, we will determine the coordinates of points when information is given with regard to length and midpoint.

(a) Determine the value(s) of *k* if the length of the line segment joining the points $A(2;-3)$ and $B(k;5)$ is $\sqrt{80}$.

(b) Determine the value(s) of *x* and *y* if the $M(1;-2)$ is the midpoint of the line joining the points $D(x;-5)$ and $E(3;y)$.

Solutions



(a) In this example you are not expected to find the length between two points but to find the coordinates of the points that will give you a certain length.
The value of *k* is the *x*-coordinate of B

$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$

Substitute the points $A(2;-3)$ and $B(k;5)$

$\therefore AB^2 = (k-2)^2 + (5-(-3))^2$

But it is given that $AB = \sqrt{80}$

$\therefore \left(\sqrt{80}\right)^2 = (k-2)^2 + (5-(-3))^2$ Substitute $AB = \sqrt{80}$

$\therefore 80 = k^2 - 4k + 4 + (8)^2$

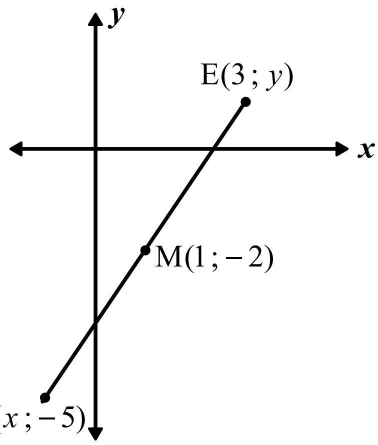
$\therefore 0 = k^2 - 4k + 4 + 64 - 80$

$\therefore 0 = k^2 - 4k - 12$ A quadratic equation is formed

$\therefore 0 = (k-6)(k+2)$ Factorise

$\therefore k = 6$ or $k = -2$

$\therefore B(-2\,;5)$ or $B(6\,;5)$



(b) State the formula:

$M(x_M;y_M) = M\left(\frac{x_E + x_D}{2};\frac{y_E + y_D}{2}\right)$

Substitute the points, $M(1;-2)$, $D(x;-5)$ and $E(3;y)$ into the formula above.

$\therefore M(1;-2) = M\left(\frac{3+x}{2};\frac{y+(-5)}{2}\right)$

Equate the *x*-coordinates and *y*-coordinates separately.

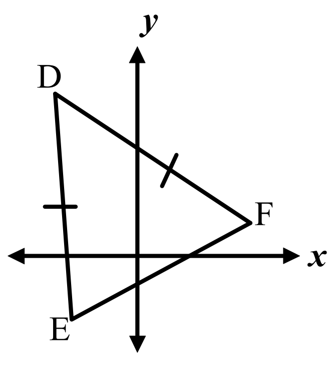
$\therefore 1 = \frac{3+x}{2}$ and $-2 = \frac{y-5}{2}$

$\therefore 2 = 3 + x$ and $-4 = y - 5$

$\therefore x = -1$ and $y = 1$

EXERCISE 1

1. In this question, it would be helpful to sketch the diagrams.
   Determine the values of *k* and *p* if:
   (a) the distance between the points $A(4\,;-2)$ and $B(k\,;-8)$ is $\sqrt{52}$ units.
   (b) $A(-3;p)$ is equidistant from the points $C(7;-1)$ and $D(4;-4)$.
2. Determine the values of *x* and *y* if it is given that:
   (a) $(-2\,;4)$ is the midpoint of the line between the points $(x\,;y)$ and $(6;-3)$.
   (b) $(-2\,;y)$ is the midpoint of the line between the points $(4\,;3)$ and $(x\,;7)$.
3. Given ΔDEF with vertices $D(-3;4)$, $E(-2;-3)$ and $F(x;1)$ with DE = DF.
   Determine the value of *x*.



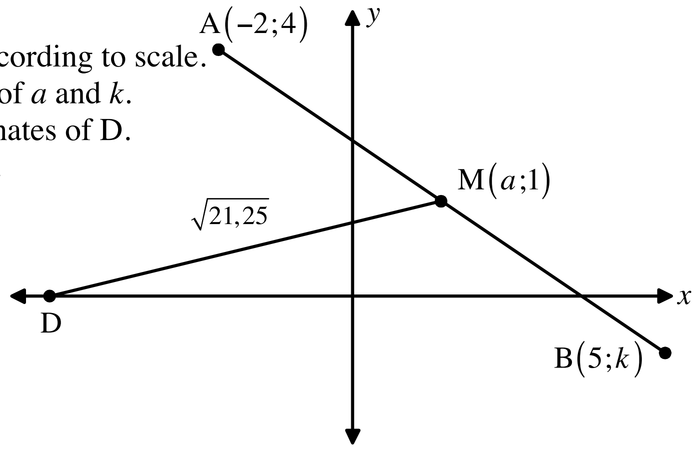
4. Show that the diagonals of the parallelogram ABCD bisect each other if the points are $A(2;3)$, $B(3;-2)$, $C(-1;0)$ and $D(-2;5)$.

5. On the Cartesian plane below $M(a;1)$ is the midpoint of line AB with $A(-2;4)$ and $B(5;k)$. Point D lies on the *x*-axis. The length of MD is $\sqrt{21,25}$.

This graph is not sketched according to scale.

(a) Determine the values of *a* and *k*.

(b) Determine the coordinates of D.
Show all calculations.



PARALLEL AND PERPENDICULAR LINES

In Grade 10 the following information regarding parallel and perpendicular lines was established:

1. If two lines AB and DC are parallel, then their gradients are equal
$\therefore \text{gradient}_{AB} = \text{gradient}_{DC}$ if $AB \parallel DC$

2. If two lines AB and DC are perpendicular, then the product of their gradients equal -1
$\therefore \text{gradient}_{AB} \times \text{gradient}_{DC} = -1$ if $AB \perp DC$

EXAMPLE 3

In each of the following determine whether $AB \parallel CD$, $AB \perp CD$ or neither.

(a) $A(-1;5)$, $B(-2;3)$, $C(9;10)$, $D(5;2)$

$$\text{Gradient}_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3-5}{-2-(-1)} = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$$

$$\text{Gradient}_{CD} = \frac{y_D - y_C}{x_D - x_C} = \frac{2-10}{5-9} = \frac{-8}{-4} = 2$$

$\therefore \text{Gradient}_{AB} = \text{Gradient}_{CD}$

$\therefore AB \parallel CD$

(b) $A(3;-3)$, $B(6;-7)$, $C(-5;0)$, $D(-1;3)$

$$\text{Gradient}_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{-7-(-3)}{6-3} = \frac{-7+3}{3} = \frac{-4}{3}$$

$$\text{Gradient}_{CD} = \frac{y_D - y_C}{x_D - x_C} = \frac{3-0}{-1-(-5)} = \frac{3}{-1+5} = \frac{3}{4}$$

$\therefore \text{Gradient}_{AB} \times \text{Gradient}_{CD} = \frac{-4}{3} \times \frac{3}{4} = -1$

$\therefore AB \perp CD$

EXAMPLE 4

Given: $A(-2;1)$, $B(k;-5)$, $C(4;6)$, $D(-2;7)$

Calculate the value of *k* in each case if:

(a) AB||CD (b) AB ⊥ CD

Solutions

(a) AB||CD

$\therefore \text{Gradient}_{AB} = \text{Gradient}_{CD}$

$\therefore \frac{y_B - y_A}{x_B - x_A} = \frac{y_D - y_C}{x_D - x_C}$

$\therefore \frac{-5-1}{k-(-2)} = \frac{7-6}{-2-4}$

$\therefore \frac{-6}{k+2} = \frac{1}{-6}$

$\therefore 36 = k+2$

$\therefore k = 34$

(b) AB ⊥ CD

$\therefore \text{Gradient}_{AB} \times \text{Gradient}_{CD} = -1$

$\therefore \frac{y_B - y_A}{x_B - x_A} \times \frac{y_D - y_C}{x_D - x_C} = -1$

$\therefore \frac{-5-1}{k-(-2)} \times \frac{7-6}{-2-4} = -1$

$\therefore \frac{-6}{k+2} \times \frac{1}{-6} = -1$

$\therefore \frac{1}{k+2} = -1$

$\therefore 1 = -k-2$

$\therefore k = -3$

COLLINEAR POINTS

Points that are **collinear** lie on the same line. The gradient between each pair of points is the same. For example, if the points A, B and C are collinear, then:

$\text{Gradient}_{AB} = \text{Gradient}_{BC} = \text{Gradient}_{AC}$

EXAMPLE 5

Show that the points A, B and C are collinear if the points are $A(2;-2)$, $B(1;1)$ and $C(-1;7)$.

Solution

$\text{Gradient}_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{1-(-2)}{1-2} = \frac{3}{-1} = -3$ $\text{Gradient}_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{7-1}{-1-1} = \frac{6}{-2} = -3$

$\therefore \text{Gradient}_{AB} = \text{Gradient}_{BC}$

Therefore A, B and C are collinear

EXERCISE 2

1. Calculate the gradients of AB and CD and in each case state whether AB and CD are:

   (1) parallel (2) perpendicular (3) neither

   (a) $A(2;-1)$, $B(5;-3)$, $C(-1;1)$, $D(-4;3)$

   (b) $A(4;2)$, $B(-1;-2)$, $C(2;0)$, $D(10;-10)$

   (c) $A(5;4)$, $B(2;7)$, $C(-7;1)$, $D(-5;3)$

   (d) $A(-1;-5)$, $B(-1;4)$, $C(-3;2)$, $D(5;2)$

2. Calculate the gradient of AB and then write down the gradient of a line perpendicular to AB.

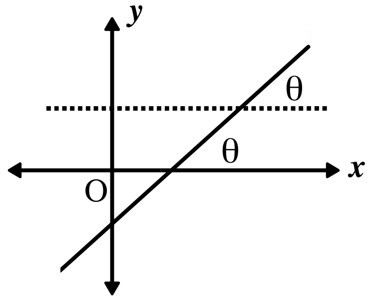
(a) $A(6;-4), B(3;1)$ (b) $A(3;1), B(-1;2)$ (c) $A(0;1), B(1;0)$

3. Calculate the value of x in each case if A, B, C and D are the points $A(3;4)$, $B(-1;7)$, $C(x;-1)$ and $D(1;8)$ and:

(a) $AB \parallel CD$ (b) $AB \perp CD$ (c) B, C and D are collinear

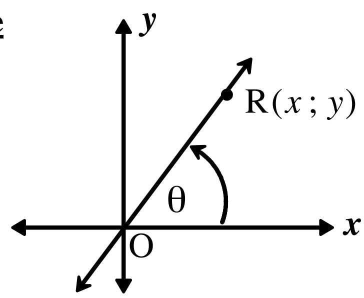
INCLINATION OF A LINE

The inclination of a line is the angle formed with the horizontal in an anti-clockwise direction. On the Cartesian plane, the inclination of a line is calculated by finding the **angle formed at the *x*-axis** measured in anti-clockwise direction. θ is the angle of inclination of line AB.



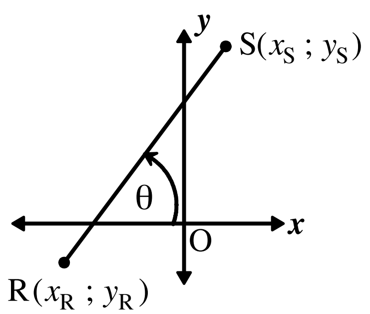
Formula for finding the angle of inclination of a line

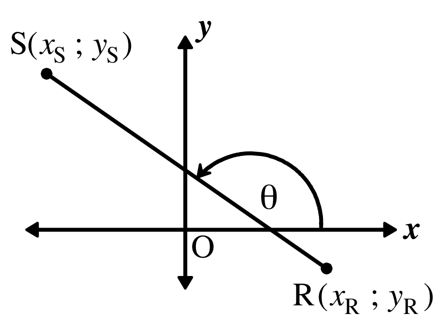
If $R(x;y)$ is a point on the terminal arm of θ, then by definition, $\tan\theta = \frac{y}{x}$. But with $O(0;0)$, the gradient of line OR $= \frac{y-0}{x-0} = \frac{y}{x}$.



$\tan\theta = \text{Gradient}_{OR}$, where θ is the angle of inclination of line OR.

Consider lines that do not pass through the origin.





$\therefore \tan\theta = \text{gradient}_{RS}$

$\therefore \tan\theta = \frac{y_S - y_R}{x_S - x_R}$

$\therefore \tan\theta = \text{gradient}_{RS}$

$\therefore \tan\theta = \frac{y_S - y_R}{x_S - x_R}$

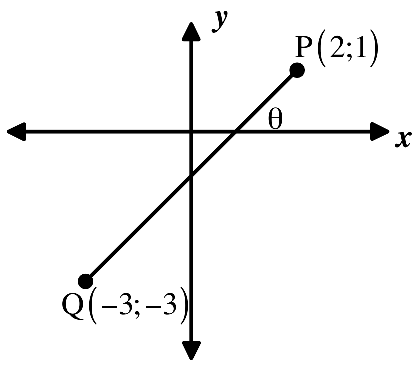
Gradient of RS is **positive** (slopes right) $\therefore \theta$ is **acute** $\therefore \theta =$ *reference angle*	Gradient of RS is **negative** (slopes left) $\therefore \theta$ is **obtuse** (lies in 2^{nd} quad) $\therefore \theta = 180° -$ *reference angle*

Note: Refer to Trigonometry Chapter 6 (page 160) which deals with reference angles.

EXAMPLE 6

Determine the inclination of the line segment in each case.

(a) $P(2;1)$ and $Q(-3;-3)$



Let θ be the angle of inclination.

$\tan\theta = \text{gradient}_{PQ}$ (State the formula)

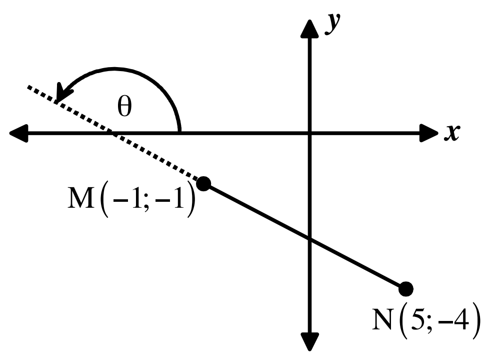
$\therefore \tan\theta = \dfrac{y_Q - y_P}{x_Q - x_P}$

$\therefore \tan\theta = \dfrac{-3-1}{-3-2}$ (Substitute the points)

$\therefore \tan\theta = \dfrac{-4}{-5} = \dfrac{4}{5}$

$\therefore \theta = 38{,}7°$ (Gradient is positive and therefore the angle of inclination is acute)

(b) $M(-1;-1)$ and $N(5;-4)$



Let θ be the angle of inclination.

$\tan\theta = \text{gradient}_{MN}$

$\therefore \tan\theta = \dfrac{y_N - y_M}{x_N - x_M}$

$\therefore \tan\theta = \dfrac{-4-(-1)}{5-(-1)}$

$\therefore \tan\theta = \dfrac{-3}{6}$

$\therefore \tan\theta = -\dfrac{1}{2}$

$\therefore \theta = 180° - 26{,}5650...°$ (Gradient is negative and therefore the angle inclination is obtuse).

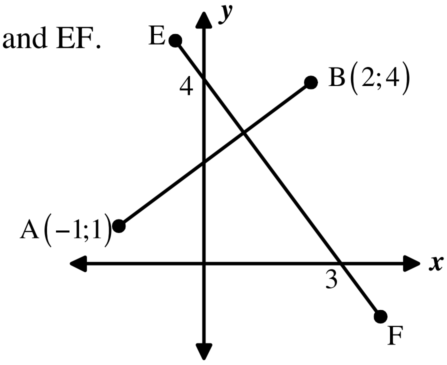
$\therefore \theta = 153{,}4°$

EXERCISE 3

1. Calculate, rounded off to two decimal places, the inclination of the line joining the points in each case.
   (a) $A(3;-1)$ and $B(-2;-3)$. (b) $C(-5;5)$ and $E(2;-4)$.
   (c) $F(4;5)$ and $G(4;9)$. (d) $H(7;-2)$ and $I(5;1)$.
   (e) $J(0;0)$ and $K(3;4)$.

2. Calculate the gradient (if possible) of a line with inclination
   (a) 135° (b) 45° (c) 60° (d) 150° (e) 90°

3. Calculate the angle of inclination of line AB and EF.

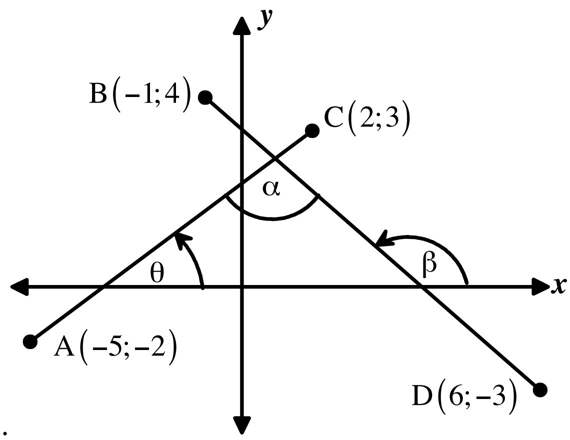


EXAMPLE 7

Given $A(-5;-2)$, $B(-1;4)$, $C(2;3)$ and $D(6;-3)$. Determine the angle α formed between lines AC and BD.

Solution

Let the inclination of AC and BD be equal to θ and β respectively.



$\tan\theta = \text{grad}_{AC}$ $\qquad$ $\tan\beta = \text{grad}_{BD}$

$\therefore \tan\theta = \dfrac{3-(-2)}{2-(-5)}$ $\qquad$ $\therefore \tan\beta = \dfrac{-3-4}{6-(-1)}$

$\therefore \tan\theta = \dfrac{5}{7}$ $\qquad$ $\therefore \tan\beta = \dfrac{-7}{7} = -1$

$\therefore \theta = 35{,}5376...^\circ$ $\qquad$ $\therefore \beta = 180^\circ - 45^\circ = 135^\circ$

Now $\alpha + \theta = \beta$ $\qquad$ Exterior angle of Δ .

$\therefore \alpha = \beta - \theta$

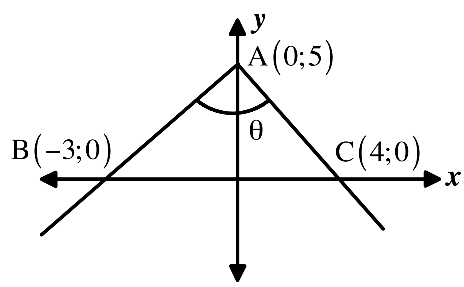
$\therefore \alpha = 135^\circ - 35{,}5376...^\circ$

$\therefore \alpha = 99{,}5^\circ$

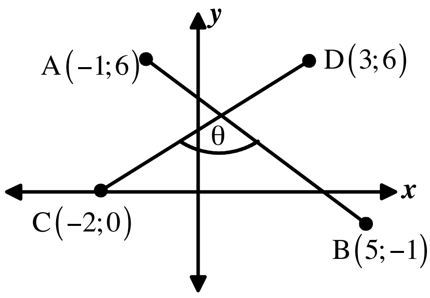
EXERCISE 4

1. Calculate the angle θ which is formed by the two lines in each of the following sketches. Round off your answers to two decimal places.

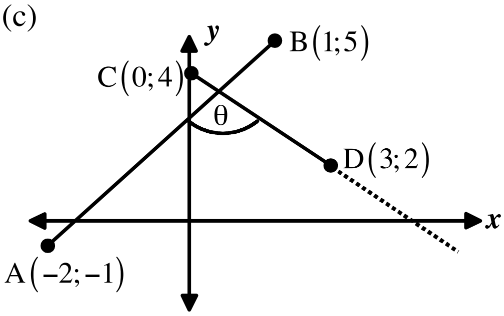
(a)



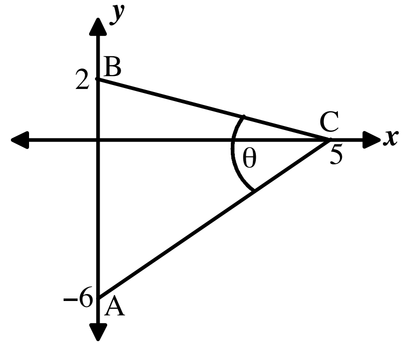
(b)



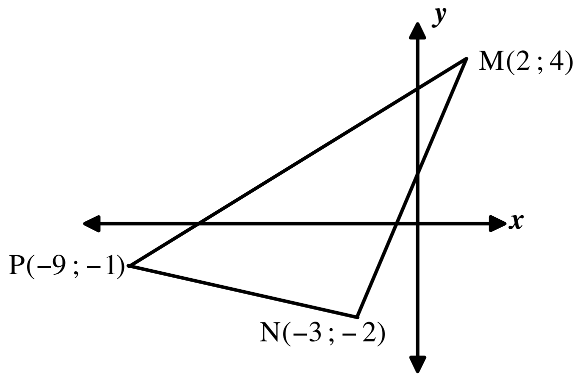
(c)



(d)



2. Refer to the diagram alongside. ΔMNP is given with $M(2;4)$, $N(-3;-2)$ and $P(-9;-1)$. Determine the size of angle M.



THE EQUATION OF A STRAIGHT LINE

Different forms of the straight line equation

(a)	$y = mx + c$	where m is the gradient and c is the y-intercept
(b)	$y - y_1 = m(x - x_1)$	where m is the gradient and $(x_1 ; y_1)$ a point on the line
(c)	$\frac{x}{a} + \frac{y}{b} = 1$	where a and b are the x-intercept and y-intercept respectively
(d)	x = number	Equation of a vertical line
(e)	y = number	Equation of a horizontal line

EXAMPLE 8

(a) Determine the equation of the line with a gradient of -2 passing through the point $(3;-4)$. (Gradient and a point on the line is given)

(b) Determine the equation of the line passing through the points $F(-4;2)$ and $G(-1;-2)$. (Line passes through two given points)

Solutions

(a) The first step is to find the gradient of the line. In this case the gradient is given.

$\therefore m = -2$

Substitute $m = -2$ into the straight line equation: $y - y_1 = m(x - x_1)$

$\therefore y - y_1 = -2(x - x_1)$

Then substitute the point $(x_1 ; y_1) = (3 ; -4)$

$\therefore y - (-4) = -2(x - 3)$

$\therefore y + 4 = -2x + 6$

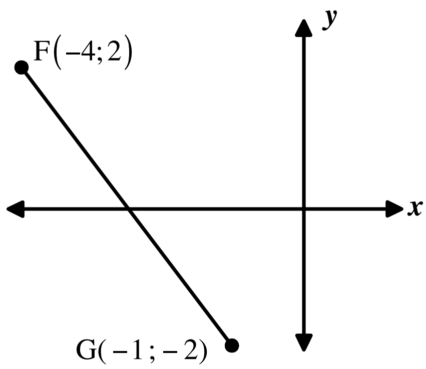
$\therefore y = -2x + 2$

(b) Determine the gradient first:

$$m_{GF} = \frac{y_G - y_F}{x_G - x_F} = \frac{-2-2}{-1-(-4)} = \frac{-4}{3} = -\frac{4}{3}$$

Substitute $m = -\frac{4}{3}$: $y - y_1 = m(x - x_1)$

$\therefore y - y_1 = -\frac{4}{3}(x - x_1)$



Now it is important to note that any of the two points that lie on the line can be substituted into the equation.

Substitute $F(-4;2)$ or Substitute $G(-1;-2)$

$\therefore y-(2)=-\frac{4}{3}(x-(-4))$	$\therefore y-(-2)=-\frac{4}{3}(x-(-1))$
$\therefore y-2=-\frac{4}{3}(x+4)$	$\therefore y+2=-\frac{4}{3}(x+1)$
$\therefore y-2=-\frac{4}{3}x-\frac{16}{3}$	$\therefore y+2=-\frac{4}{3}x-\frac{4}{3}$
$\therefore y=-\frac{4}{3}x-\frac{10}{3}$	$\therefore y=-\frac{4}{3}x-\frac{10}{3}$

Note: The point you substitute into the equation must be a point on the line.

EXERCISE 5

1. Determine the equation of the line if the gradient and a point on the line are given.

   (a) $-3;\ (-1;2)$ (b) $\frac{2}{3};\ (6;-4)$ (c) $-1;\ \left(\frac{1}{2};-5\right)$

   (d) undefined; $(-8;2)$ (e) $0;\ (-2;7)$

2. Determine the equation of the line through the two given points:

   (a) $(-1;-7)$ and $(-2;-5)$ (b) $(3;2)$ and $(1;6)$

   (c) $(5;0)$ and $(-3;2)$ (d) $(-1;6)$ and $(-1;-2)$

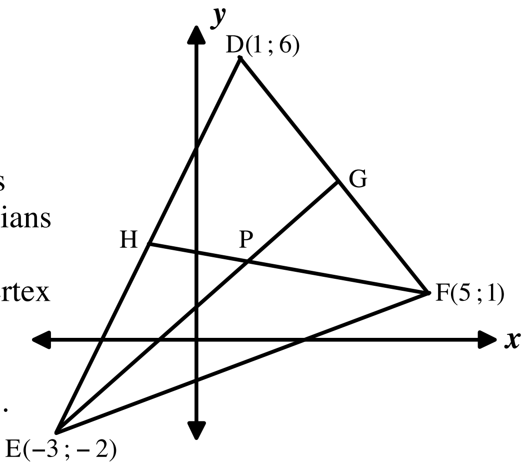
   (e) $(-5;3)$ and $(1;3)$ (f) $(0,5;4)$ and $(-2;8)$

3. In the diagram below, the vertices of ΔDEF are $D(1;6)$, $E(-3;-2)$ and $F(5;1)$.

   (a) Determine the coordinates of the midpoints H and G of DE and DF respectively.

   (b) Determine the equation of lines EG and FH which are two medians of the triangle. (A median is a line from the vertex of a triangle to the midpoint of the opposite side).

   (c) Determine the coordinates of P.



MORE ON PARALLEL AND PERPENDICULAR LINES

EXAMPLE 9

(a) Determine the equation of the line that is parallel to $3y-2x=6$ and passes through the point $(9;-1)$

(b) Determine the equation of the line that is perpendicular to $y=\frac{1}{2}x+1$ and passes through the point $(-6;2)$

Solutions

(a) Firstly, it is important to note that an equation has been given. Refer to this equation as the "OLD" equation. In this example, $3y - 2x = 6$ is the "OLD' equation. Refer to equation you are required to find as the "NEW" equation. Rewrite the "OLD" equation in the standard form $y = mx + c$.

$\therefore 3y = 2x + 6$

$\therefore y = \frac{2}{3}x + 2$

$\therefore m_{\text{old}} = \frac{2}{3}$

$\therefore m_{\text{new}} = \frac{2}{3}$ (OLD line ‖ NEW line)

"NEW" equation: $y - y_1 = m_{new}(x - x_1)$

$\therefore y - y_1 = \frac{2}{3}(x - x_1)$ Substitute m_{new}

$\therefore y - (-1) = \frac{2}{3}(x - 9)$ Substitute the point $(9;-1)$

$\therefore y + 1 = \frac{2}{3}x - 6$

$\therefore y = \frac{2}{3}x - 7$

(b) Refer to the equation $y = \frac{1}{2}x + 1$ as the "OLD" equation.

Refer to the equation that you are required to find as the "NEW" equation.

The "OLD" equation is in standard form $y = \frac{1}{2}x + 1$ and $m_{\text{old}} = \frac{1}{2}$

$m_{\text{old}} \times m_{\text{new}} = -1$ (OLD line $\perp$ NEW line)

$\therefore \frac{1}{2} \times m_{\text{new}} = -1$

$\therefore m_{\text{new}} = -2$

"NEW" equation: $y - y_1 = m_{\text{new}}(x - x_1)$

$\therefore y - y_1 = -2(x - x_1)$ Substitute m_{new} in first

$\therefore y - 2 = -2(x - (-6))$ Substitute the point $(-6;2)$

$\therefore y - 2 = -2(x + 6)$

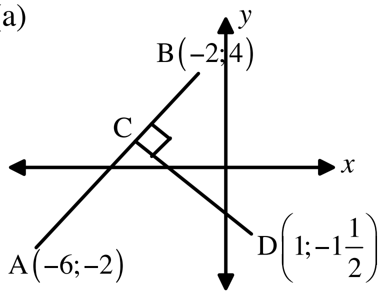
$\therefore y - 2 = -2x - 12$

$\therefore y = -2x - 10$

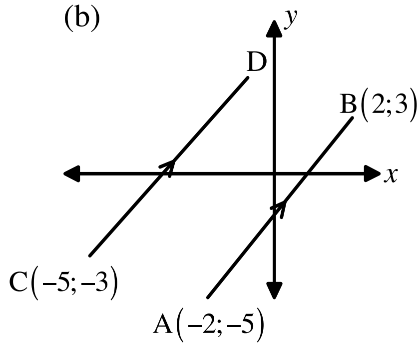
EXERCISE 6

1. Determine the equation of the line:
   (a) passing through the point $(-4;3)$ and perpendicular to $2y = 3x + 6$.
   (b) passing through the point $(-8;-1)$ and parallel to $x - 2y + 2 = 0$.
   (c) parallel to $2y - x = 4$ and passing through $(-1;-2)$.
   (d) perpendicular to $3x - y = 4$ and passing through $(6;4)$.
   (e) through the point $(-2;5)$ with inclination of $135°$.
   (f) through the point $(0;-4)$ with inclination of $60°$.
2. Determine the equation of the line:
   (a) perpendicular to the y-axis and going through the point $(2;-2)$
   (b) parallel to the y-axis and going through the point $(2;-2)$
3. Determine the equation of CD in each case.

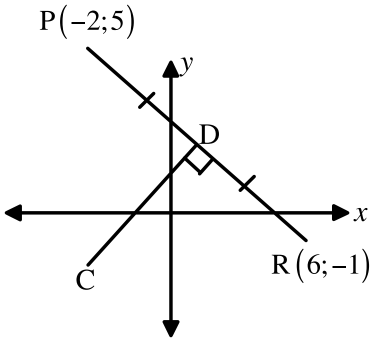
(a)



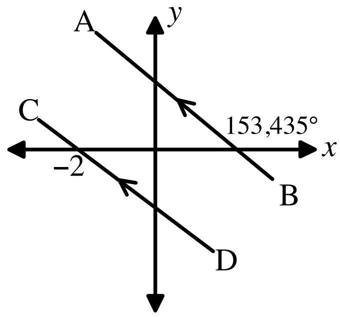
(b)



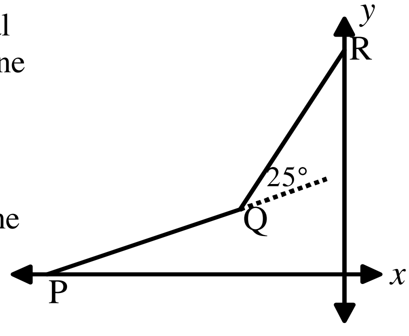
(c)



(d)

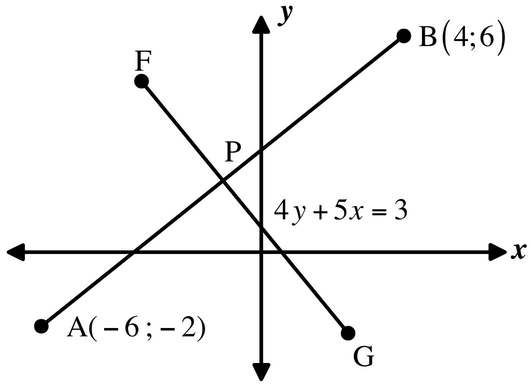


4. The line PQ's inclination with the horizontal is increased by 25° at Q. The equation of line RQ is $y - \frac{3}{2}x = 2$.
   (a) Calculate the inclination of line RQ.
   (b) Find the gradient of PQ correct to one decimal place.



INTERSECTING LINES

EXAMPLE 10



Consider the lines AB and line FG with equation $4y+5x=3$.

(a) Determine the equation of line AB.

(b) Determine the coordinates of P.

(c) Show that line FG is perpendicular to line AB and bisects line AB.

Solutions

(a) $\text{Gradient}_{AB} = \dfrac{6-(-2)}{4-(-6)} = \dfrac{8}{10} = \dfrac{4}{5}$

$\therefore$ Equation of line AB: $y - y_1 = \dfrac{4}{5}(x - x_1)$

$\therefore y - 6 = \dfrac{4}{5}(x-4)$ Substitute $B(4;6)$:

$\therefore y - 6 = \dfrac{4}{5}x - \dfrac{16}{5}$

$\therefore y = \dfrac{4}{5}x + \dfrac{14}{5}$

(b) Write the given equation in standard form:

$4y + 5x = 3$

$\therefore 4y = -5x + 3$

$\therefore y = -\dfrac{5}{4}x + \dfrac{3}{4}$

At P the two lines intersect. Therefore we have to solve the two equations simultaneously.

$\dfrac{4}{5}x + \dfrac{14}{5} = -\dfrac{5}{4}x + \dfrac{3}{4}$

$\therefore 16x + 56 = -25x + 15$ LCD is 20 (multiply each term by 20)

$\therefore 16x + 25x = 15 - 56$

$\therefore 41x = -41$

$\therefore x = -1$

$\therefore y = \dfrac{4}{5}(-1) + \dfrac{14}{5} = 2$

$\therefore P(-1;2)$

(c) In order to show that the two lines are perpendicular, show that the product of their gradients is equal to -1.

$\text{Gradient}_{FG} = -\dfrac{5}{4}$ and $\text{Gradient}_{AB} = \dfrac{4}{5}$

$\therefore \text{Gradient}_{FG} \times \text{Gradient}_{AB} = -1$

$\therefore FG \perp AB$

In order to show that FG bisects line AB, show that $P(-1;2)$ is the midpoint of AB.

Midpoint of AB: $\left(\frac{4+(-6)}{2};\frac{6+(-2)}{2}\right)=(-1\,;2)$

The coordinates of P are $(-1\,;2)$

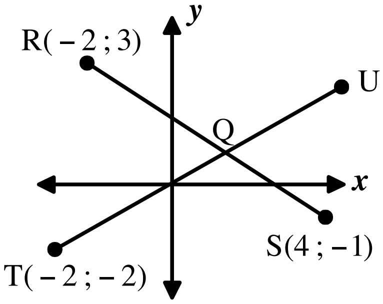
Therefore, P is the midpoint of AB.

EXERCISE 7

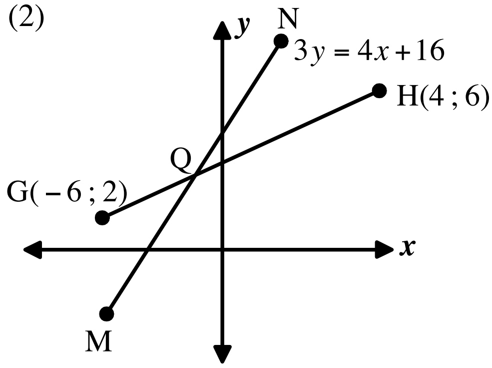
(a) Determine the coordinates of the point of intersection of the lines $y-3x=3$ and $2y+4x=16$

(b) Determine the coordinates of Q, the point of intersection in each case.

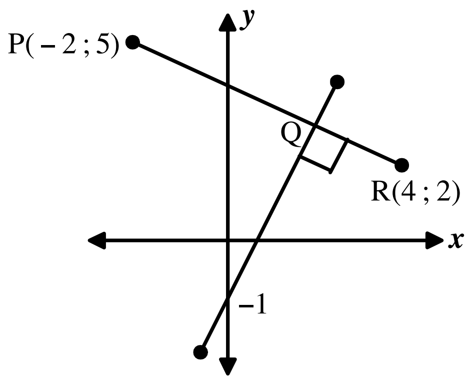
(1)



(2)

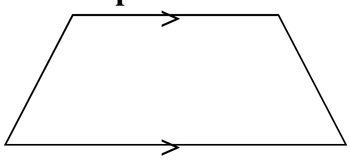


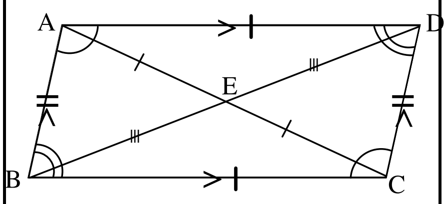
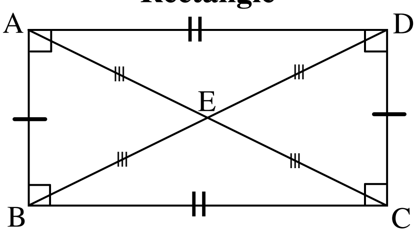
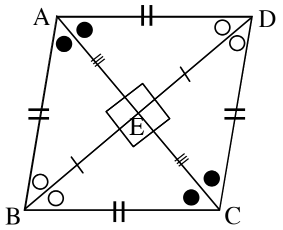
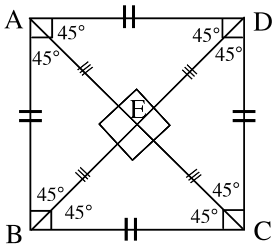
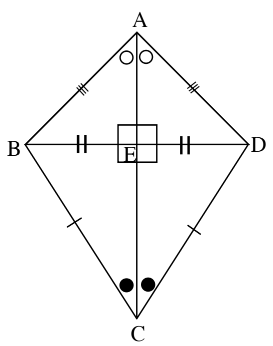
(3)



APPLICATION TO QUADRILATERALS

The following is a summary of all the properties of the different type of quadrilaterals.

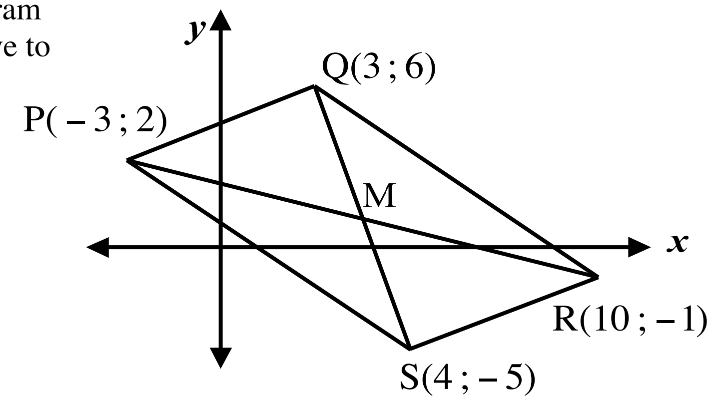
Quadrilateral	Definition	Properties
Trapezium 	A **trapezium** is a quadrilateral with one pair of opposite sides parallel.	• One pair of sides parallel

Parallelogram 	A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel	• Two pairs of opposite sides equal and parallel • Opposite angles equal • Diagonals bisect
Rectangle 	A **rectangle** is a parallelogram with all interior angles equal to $90°$.	• All the properties of a parallelogram AND • Diagonals are equal in length • Interior angles are right angles
Rhombus 	A **rhombus** is a parallelogram with equal sides	• All the properties of a parallelogram AND • Diagonals bisect at right angles. • Diagonals bisect the opposite angles • All sides are equal in length
Square 	A **square** is a rectangle with equal sides.	• All the properties of a rectangle AND • All sides equal in length. • Diagonals bisect each other at right angles. • Diagonals bisect interior angles (each bisected angle equals $45°$)
Kite 	A **kite** is a quadrilateral with two pairs of adjacent sides equal in length	• Adjacent pairs of sides are equal in length • The longer diagonal bisects the opposite angles. • The longer diagonal bisects the other diagonal. • The diagonals intersect at right angles.

EXAMPLE 11

Use analytical methods to show that PQRS is a parallelogram if $P(-3;2)$, $Q(3;6)$, $R(10;-1)$ and $S(4;-5)$.

Solution



For PQRS to be a parallelogram the diagonals PR and QS have to bisect.
This means that the two diagonals have the same midpoint.

Midpoint of QS:

$$\left(\frac{x_S + x_Q}{2};\frac{y_S + y_Q}{2}\right)$$

$$=\left(\frac{4+3}{2};\frac{(-5)+6}{2}\right)=\left(\frac{7}{2};\frac{1}{2}\right)$$

Midpoint of PR:

$$\left(\frac{x_R + x_P}{2};\frac{y_R + y_P}{2}\right)$$

$$=\left(\frac{10+(-3)}{2};\frac{-1+2}{2}\right)=\left(\frac{7}{2};\frac{1}{2}\right)$$

$\therefore$ Midpoint of QS $=$ Midpoint of PR $=$ the point M
$\therefore$ PQRS is a parallelogram (Diagonals bisect)

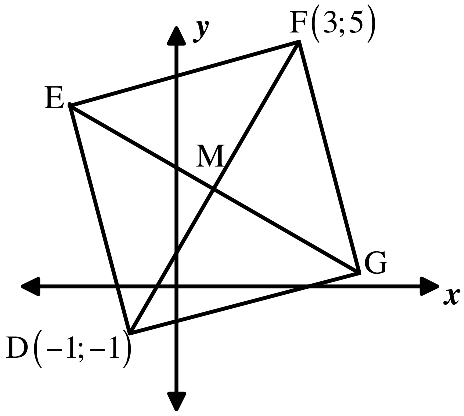
We could have also proven PQRS a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$ or $PQ = RS$ and $PS = QR$ as well.
The method used above is simpler and faster.

EXAMPLE 12

DEFG forms a rhombus with $D(-1;1)$ and $F(3;5)$ being two of the vertices.

(a) Find the midpoint of diagonal DF.
(b) Determine the equation of diagonal EG.

Solutions



(a) $$M\left(\frac{x_F + x_D}{2};\frac{y_F + y_D}{2}\right)$$

$$= M\left(\frac{3+(-1)}{2};\frac{5+(-1)}{2}\right)$$

$$= M\left(\frac{2}{2};\frac{4}{2}\right)$$

$$= M(1;2)$$

(b) EG ⊥ DF ….. diagonals of a rhombus bisect at right angles.

$\therefore m_{DF} \times m_{EG} = -1$

$m_{DF} = \frac{y_F - y_D}{x_F - x_D} = \frac{5-(-1)}{3-(-1)} = \frac{6}{4} = \frac{3}{2}$

$\therefore \frac{3}{2} \times m_{EG} = -1$ and

$\therefore m_{EG} = \frac{-2}{3}$

Equation of a straight line: $y - y_1 = m_{EG}(x - x_1)$

$\therefore y - y_1 = \frac{-2}{3}(x - x_1)$ Substitute gradient first

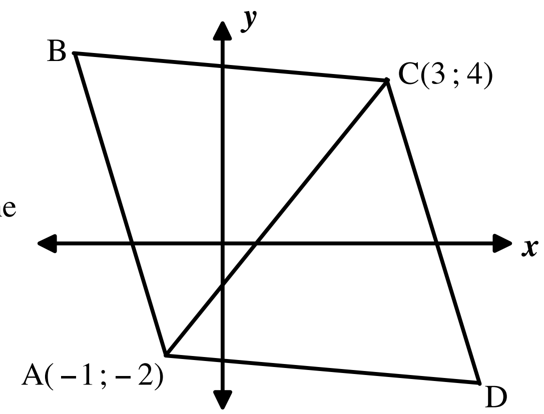
$\therefore y - 2 = \frac{-2}{3}(x - 1)$ Substitute the point $(1;2)$ (lies on the line)

$\therefore y - 2 = \frac{-2}{3}x + \frac{2}{3}$

$\therefore y = \frac{-2}{3}x + \frac{8}{3}$

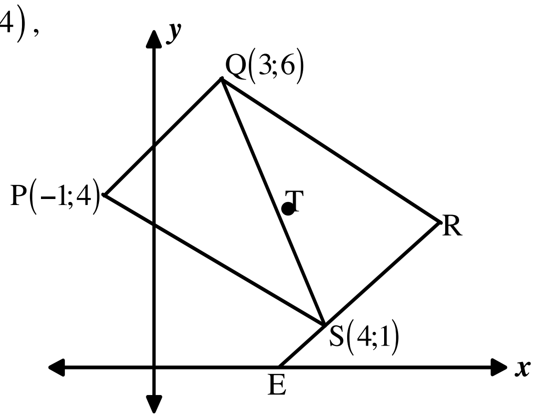
EXERCISE 8

1. Quadrilateral DEFG is formed by the points $D(-5;3)$, $E(3;5)$, $F(2;1)$ and $G(-6;-1)$. Now answer the following questions.
   (a) State 3 properties of a parallelogram.
   (b) Use **three** different methods to show that DEFG is a parallelogram.
2. $M(-3;2)$, $N(3;6)$, $O(9;-2)$ and $P(3;-6)$ are the points of quadrilateral MNOP. Show that:
   (a) MNOP is a parallelogram. (b) MNOP is not a rectangle.
3. $A(-2;3)$, $B(x;y)$, $C(1;4)$ and $D(-1;2)$ are the vertices of parallelogram ABCD. Find $B(x;y)$.
4. $L(-1;-1)$, $M(-2;4)$, $N(x;y)$ and $P(4;0)$ are the vertices of parallelogram LMNP.
   (a) Determine the coordinates of N.
   (b) Show that $MP \perp LN$ and state what type of quadrilateral LMNP is other than a parallelogram.
   (c) Show that LMNP is a square.
5. Given: ABCD is a rhombus with $A(-1;-2)$ and $C(3;4)$.
   (a) Determine the equation of AC.
   (b) Determine the equation of BD.
   (c) If D is the point $D(6;y)$, determine the value of y and hence the coordinates of B.



6. PQRS is a parallelogram with $P(-1;4)$, $Q(3;6)$ and $S(4;1)$. Determine:

(a) the gradient of PQ.
(b) the midpoint T of QS.
(c) the coordinates of R.
(d) the equation of RS.
(e) the inclination of the line PE if E is the x-intercept of the line RS produced.
(f) the size of $P\hat{E}S$

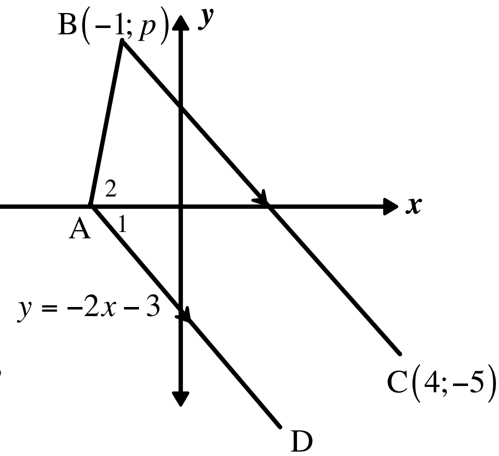


REVISION EXERCISE

1. Refer to the diagram alongside

Line AD is defined by the equation $y=-2x-3$.
Points $B(-1;p)$ and $C(4;-5)$ are placed in such a way that BC||AD. Point A lies on the x-axis

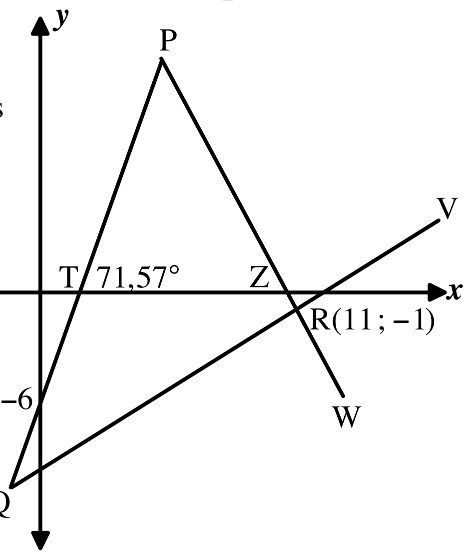
(a) Determine the value of p.
(b) Calculate the magnitude of $D\hat{A}B$



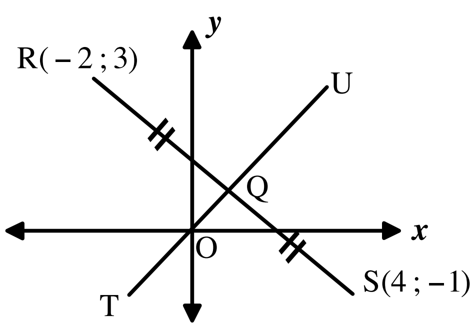
2. In the diagram alongside PW intersects QV at $R(11;-1)$. T and Z are x-intercepts of PQ and PW respectively. PQ intersects the y-axis at -6. $P\hat{T}Z=71{,}57°$. The equation of PW is given by $y+x=10$. $PW \perp QV$

Determine:

(a) the gradient of PQ.
(b) the equation of PQ.
(c) the coordinates of P.
(d) the size of angle $Q\hat{P}W$
(e) the equation of QV.



3. Determine the equation of line TU.

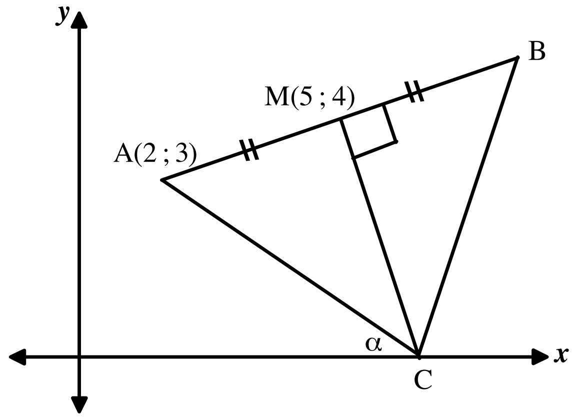


4. The straight line with equation $3y - x = 11$ is parallel to the straight line passing through $A(-4;1)$ and $B(k+2;4)$. Calculate the value of k.

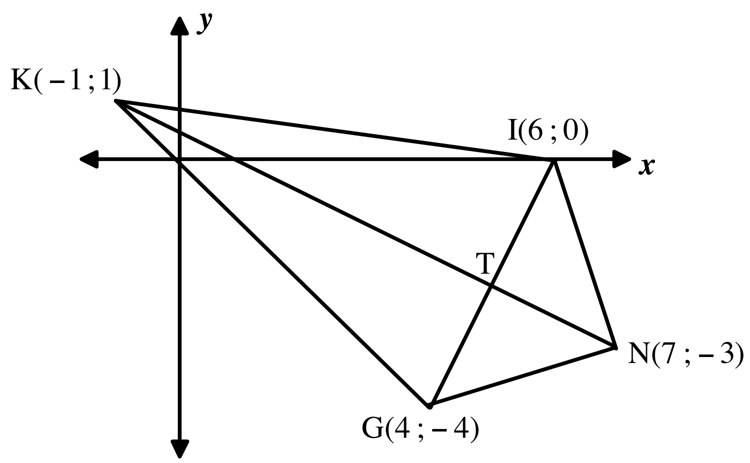
5. Determine the equation of a line with inclination $120°$ and passing through the point $(\sqrt{3};-1)$.

6. Consider the points $A(-1;1)$, $B(p;p^2)$ and $C(1;2p-1)$.
   (a) Show that these three points are collinear.
   (b) Find the equation of the straight line ABC, if $p = 3$.

7. In the following sketch AMB is a straight line with $AM = MB$ and $MC \perp AB$.
   (a) Calculate the coordinates of B.
   (b) Find the equation of MC.
   (c) Find the size of α correct to one decimal place.

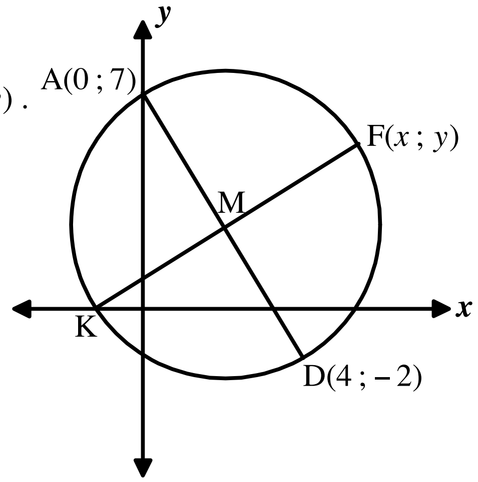


8. In the diagram below, a kite, KING, is drawn
   (a) Calculate the lengths of KI and GN and leave your answers in simplest surd form.
   (b) Prove, using analytical methods, that $IG \perp KN$.
   (c) (1) Calculate the angle of inclination of KG
       (2) Hence, or otherwise calculate the size of $G\hat{K}I$.
   (d) Calculate the area of ΔGKI.
   (e) Determine the equation of line KN.
   *(f) If ΔGIN is reflected about the line GI, $\Delta GIN'$ is formed. Determine the coordinates of N'.

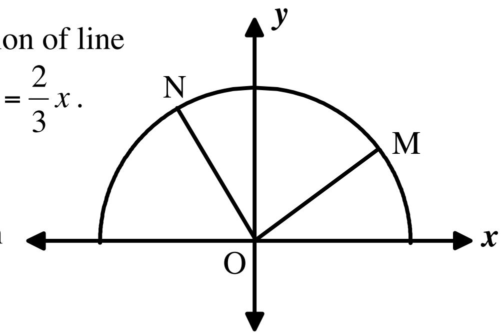


SOME CHALLENGES

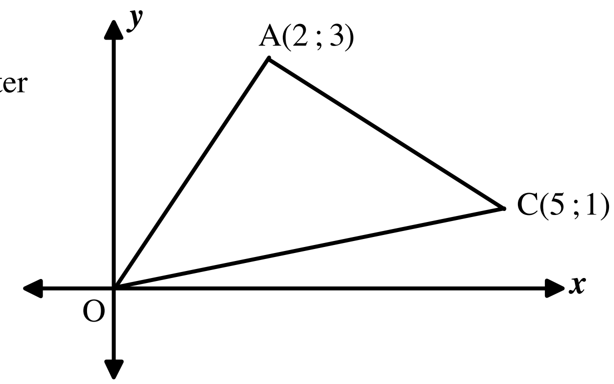
1. A circle with centre M passes through the points $A(0;7)$, $D(4;-2)$ and $F(x;y)$. Determine the coordinates of F. Show all working out.



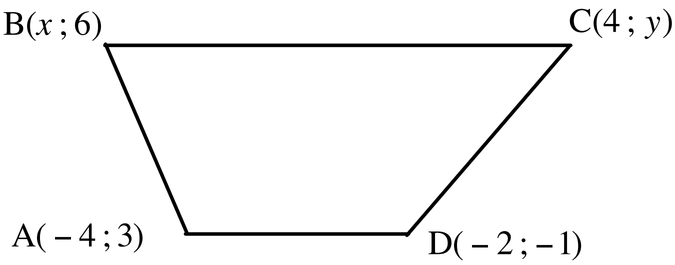
2. A semi-circle with centre O is given. Lines ON and OM are drawn. The equation of line ON is $y = -\frac{3}{2}x$ the equation of OM is $y = \frac{2}{3}x$.
   (a) State why we may conclude that $ON \perp OM$.
   (b) Determine the angle of inclination of line OM.
   (c) Determine the gradient of NM.



3. Calculate the area of ΔAOC. (Attempt this question only after studying Chapter 6)



4. The coordinates of trapezium ABCD are $A(-4;3)$, $B(x;6)$, $C(4;y)$ and $D(-2;-1)$. $AD \parallel BC$ and $BC = 2AD$. Determine the values of x and y.



CHAPTER 5 – FUNCTIONS

REVISION OF GRADE 10 FUNCTIONS

SUMMARY OF GRADE 10 LINEAR FUNCTIONS

(a) The general equation for the straight line graph is $y = ax + q$ (remember that the exponent of x is always equal to 1).

(b) Draw the "mother graph" $y = ax$ by using a table of selected x-values. Make y the subject of the formula if necessary.

(c) The value of a is called the slope or the gradient of the line.
If $a > 0$, the line slopes up from left to right.
As the positive value of a increases, the straight line gets steeper (closer to the y-axis).
If $a < 0$, the line slopes down from left to right.
As the negative value of a decreases, the straight line gets steeper (closer to the y-axis).

(d) Draw the graph of $y = ax + q$ by translating the "mother graph".
If $q > 0$, then the graph of $y = ax$ shifts q units upwards.
If $q < 0$, then the graph of $y = ax$ shifts q units downwards.

(e) The y-intercept of the graph of $y = ax + q$ is the value of q.
The y-intercept can also be determined by letting $x = 0$ in the equation of $y = ax + q$.

(f) The x-intercept can be determined by letting $y = 0$ in the equation of $y = ax + q$.

(g) Linear functions of the form $bx + cy = d$ can be sketched by using the dual-intercept method:
y-intercept: Let $x = 0$
x-intercept: Let $y = 0$

(h) Horizontal lines have equations of the form $y = \text{number}$.
The gradient of a horizontal line is zero.
Vertical lines have equations of the form $x = \text{number}$.
The gradient of a vertical line is undefined.

(i) If $a > 0$ then the graph of the straight line increases for all values of x.
If $a < 0$ then the graph of the straight line decreases for all values of x.

(j) For any linear function $y = ax + q$:
Domain: $x \in (-\infty\,;\,\infty)$ Range: $y \in (-\infty\,;\,\infty)$

(k) Methods of determining the equation of a given line in the form $y = ax + q$
   (1) If you are given the y-intercept and another point on the line, then let q be the value of the y-intercept and then substitute the point to get the value of a.
   (2) If you are given two points on the line, determine the value of a (the gradient) and hence substitute a point to calculate the value of q.

(l) Remember that parallel lines have equal gradients and the product of the gradients of perpendicular lines is equal to -1.

(m) Use simultaneous equations to determine the coordinates of the point of intersection of two lines.

SUMMARY OF GRADE 10 QUADRATIC FUNCTIONS (PARABOLAS)

(a) The general equation for the quadratic function is $y = ax^2 + q$.
(remember that the exponent of x is always squared).

(b) Draw the "mother graph" $y = ax^2$ by using a table of selected x-values.

(c) The value of a tells us if the graph is concave (happy) or convex (sad).
If $a > 0$, the shape is concave $\cup$ If $a < 0$, the shape is convex. $\cap$

As the positive value of a increases, the arms gets closer to the y-axis.
As the negative value of a decreases, the arms get closer to the y-axis.

(d) Draw the graph of $y = ax^2 + q$ by translating the "mother graph".
If $q > 0$, then the graph of $y = ax^2$ shifts q units upwards.
If $q < 0$, then the graph of $y = ax^2$ shifts q units downwards.

(e) The y-intercept of the graph of $y = ax^2 + q$ is the value of q.
The y-intercept can also be determined by letting $x = 0$ in the equation of $y = ax^2 + q$.

(f) If $a > 0$ the parabola has a minimum value at q.
If $a < 0$ the parabola has a maximum value at q.

(g) If the parabola cuts the x-axis, the x-intercepts can be determined by letting $y = 0$ in the equation of $y = ax^2 + q$ and solving for x.

(h) The equation of the axis of symmetry for the graph of a parabola of the form $y = ax^2 + q$ is $x = 0$ (the y-axis).

(i) If $a > 0$ then the graph of the parabola decreases for all $x < 0$ and increases for all $x > 0$.
If $a < 0$ then the graph of the parabola increases for all $x < 0$ and decreases for all $x > 0$.

(j) For any quadratic function of the form $y = ax^2 + q$:
Domain: $x \in (-\infty\,;\infty)$
Range: $y \in [q\,;\infty)$ if $a > 0$ $\quad y \in (-\infty\,;q]$ if $a < 0$

(k) Methods of determining the equation of a given parabola:
(1) If you are given the y-intercept and another point on the parabola, then in the equation $y = ax^2 + q$, let q be the value of the y-intercept and then substitute the point to get the value of a.
(2) If you are given the x-intercepts and another point, substitute these values into the equation $y = a(x - x_1)(x - x_2)$ where x_1 and x_2 are the x-intercepts. Then substitute the other point to calculate the value of a.

SUMMARY OF GRADE 10 HYPERBOLIC FUNCTIONS

(a) The general equation for the hyperbola is $y = \dfrac{a}{x} + q$ where $x \neq 0$
(remember that x is in the denominator).

(b) The hyperbola has **two asymptotes**:
Vertical asymptote: $x = 0$ Horizontal asymptote: $y = q$

(c) Draw the "mother graph" $y = \frac{a}{x}$ by using a table of selected x-values (preferably two positive and two negative factors of a).

(d) Then shift the four "mother" graph points q units up or down:

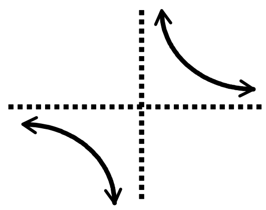
If $q > 0$, then the graph of $y = \frac{a}{x}$ shifts q units upwards.

If $q < 0$, then the graph of $y = \frac{a}{x}$ shifts q units downwards.

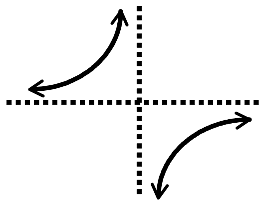
(e) Make sure that you draw the horizontal asymptote $y = q$.
Remember the the vertical asymptote is the line $x = 0$ (the y-axis).

(f) Determine the x-intercept of the graph of $y = \frac{a}{x} + q$ by letting $y = 0$.

(g) If $a > 0$ then the graph decreases for all $x < 0$ or $x > 0$.



If $a < 0$ then the graph increases for all $x < 0$ or $x > 0$.



(h) For any hyperbolic function of the form $y = \frac{a}{x} + q$:

Domain: $x \in (-\infty\,;\infty)$ where $x \neq 0$

Range: $y \in (-\infty\,;\infty)$ where $y \neq q$

(i) Method of determining the equation of a given hyperbola in the form $y = \frac{a}{x} + q$:
The value of q is obtained from the horizontal asymptote $y = q$.
Then substitute another point on the hyperbola to get the value of a.

SUMMARY OF GRADE 10 EXPONENTIAL FUNCTIONS

(a) The general equation for the exponential function is $y = ab^x + q$ where $b > 0$ and $b \neq 1$ (remember that the exponent is x).

(b) The exponential graph has **one asymptote**:
Horizontal asymptote: $y = q$

(c) Draw the "mother graph" $y = ab^x$ by using a table of selected x-values (best to use $x \in \{-1\,;0\,;1\}$).

(d) Then shift the three "mother" graph points q units up or down:

If $q > 0$, then the graph of $y = ab^x$ shifts q units upwards.

If $q < 0$, then the graph of $y = ab^x$ shifts q units downwards.

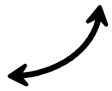
(e) Make sure that you draw the horizontal asymptote $y = q$.

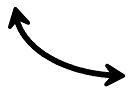
(f) Determine the x-intercept of the graph of $y = ab^x + q$ by letting $y = 0$.

(g) The y-intercept can be determined by letting $x = 0$ in the equation of $y = ab^x + q$.

(h) The value of b tells us if the graph is increasing or decreasing ($a > 0$):

If $b > 1$, the shape is increasing   If $0 < b < 1$, the shape is decreasing.





The graphs increase or decrease for all values of x.

(i) For any exponential function of the form $y = ab^x + q$ [$a > 0$, $b > 0$ and $b \neq 1$]

Domain: $x \in (-\infty\ ; \infty)$

Range: $y \in (q\ ; \infty)$

(j) Method of determining the equation of a given exponential function in the form $y = ab^x + q$:

The value of q is obtained from the horizontal asymptote $y = q$.

Then substitute another point on the graph to get the value of b.

Remember that the value of b can only be a fraction between 0 and 1 or a number greater than 1. The value of b can never be negative, 0 or 1.

SUMMARY OF GRADE 10 FUNCTIONAL NOTATION

Consider $f(x) = 3x$.

This is read as "f of x is equal to $3x$".

The symbol $f(x)$ is used to denote the element of the range to which x maps. In other words, the y-values corresponding to the x-values are given by $f(x)$, i.e. $y = f(x)$.

For example, if $x = 4$, then the corresponding y-value is obtained by substituting $x = 4$ into $3x$.

For $x = 4$, the y-value is $f(4) = 3(4) = 12$.

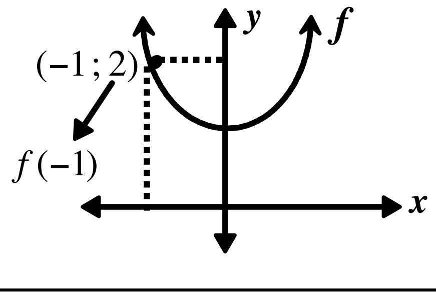
The brackets in the symbol $f(4)$ do not mean f times 4, but rather the y-value at $x = 4$.

For example, suppose that you are given the function $f(x) = x^2 + 1$ and you are required to determine $f(-1)$.

Clearly $f(-1) = (-1)^2 + 1 = 2$.

The notation $f(-1) = 2$ is simply another way of representing the point $(-1\,;2)$ which lies on the graph of $y = f(x) = x^2 + 1$.

For $x = -1$, the corresponding y-value is 2.



<u>EXERCISE 1</u> (REVISION OF GRADE 10 FUNCTIONS)

1. Draw a neat sketch graph of each of the following on separate axes.

(a) $f(x) = 2x$ (b) $g(x) = 2 + x$ (c) $h(x) = 2 - 2x$

(d) $f(x) = 2x^2$ (e) $g(x) = 2x^2 + 2$ (f) $h(x) = -2x^2 + 8$

(g) $f(x) = \frac{2}{x} + 2$ (h) $g(x) = 2^x - 2$ (i) $2x - 3y = 6$

(j) $-2x + 6 = 0$ (k) $2y - 8 = 0$ (l) $y = \frac{x^2}{2} - 2$

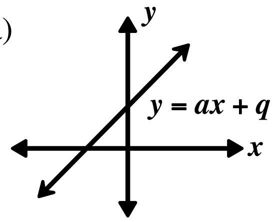
(m) $f(x) = \frac{2 - x}{x}$ (n) $g(x) = \frac{2 - x}{2}$ (o) $h(x) = 1 + \left(\frac{1}{2}\right)^x$

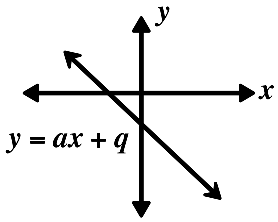
2. Given the functions:

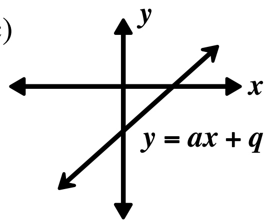
$f(x) = \frac{x}{2} + 1$ $g(x) = \frac{2}{x} + 1$ $h(x) = 2^x + 1$

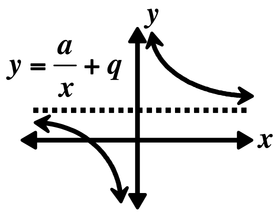
(a) Determine the coordinates of the *x*-intercept and *y*-intercept for the graph of *f*.
(b) Which of the graphs share a common horizontal asymptote?
(c) Which graph has a vertical asymptote and what is the equation of this asymptote?
(d) On the same set of axes, draw neat sketch graphs of these functions. Indicate the coordinates of the intercepts with the axes as well as the asymptotes.
(e) State the domain of g.
(f) State the range of h.
(g) Which of the functions increase for all values of x?

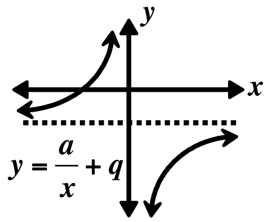
3. For each of the functions below, state whether the sign of a, b and q is positive, negative or zero.

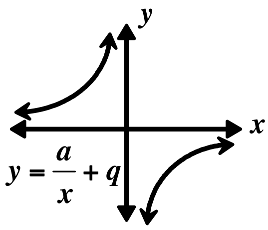
(a)


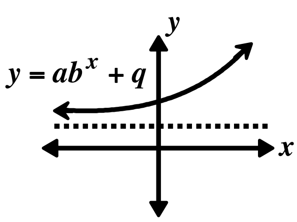
(b)


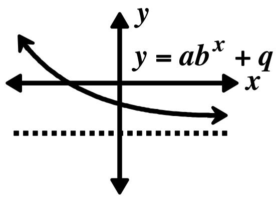
(c)


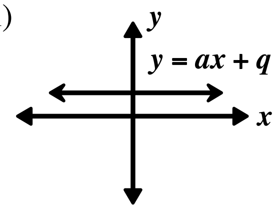
(d)


(e)


(f)


(g)


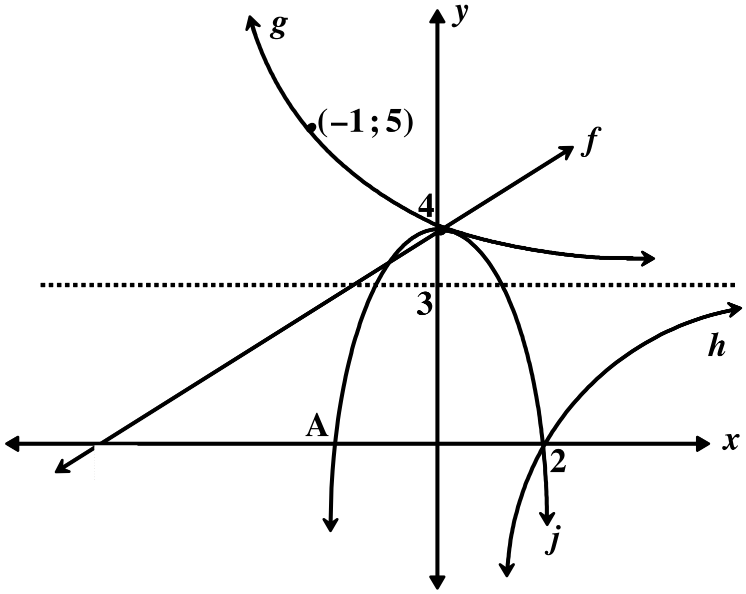
(h)


(i)


4. Draw a rough sketch graph of $y = ax^2 + q$ in each of the following cases:

(a)	$a > 0$ and $q > 0$	(b)	$a < 0$ and $q > 0$
(c)	$a > 0$ and $q < 0$	(d)	$a < 0$ and $q < 0$
(e)	$a > 0$ and $q = 0$	(f)	$a < 0$ and $q = 0$

5. In the diagram below, the graphs of four functions are drawn. The graph of f cuts the y-axis at 4 and the x-axis at -8. The graph of g cuts the y-axis at 4 and shares a common horizontal asymptote with the graph of h. The graph of h cuts the x-axis at 2. The graph of j cuts the y-axis at 4 and the x-axis at 2. A represents the other x-intercept for the graph of j. The point $(-1 ; 5)$ lies on the graph of g.



(a) Determine the equation of f in the form $y = ax + q$.

(b) Determine the equation of g in the form $y = b^x + q$.

(c) Determine the equation of h in the form $y = \frac{a}{x} + q$.

(d) Write down the coordinates of A.

(e) Determine the equation of j in the form $y = ax^2 + q$.

(f) Determine the domain and range of all four graphs.

(g) For which values of x does the graph of j decrease?

(h) If the graph of j is reflected about the x-axis, determine the equation of the newly formed graph.

(i) If the graph of g is reflected about the y-axis, determine the equation of the newly formed graph.

(j) If $h(x) = -x^2 + 4$, determine $h(2x) - 6$ and sketch the graph of $p(x) = h(2x) - 6$.

6. The following information is given for the graph of $y = ax^2 + q$:

$a < 0$ $\quad f(1) = -3$ $\quad$ maximum value $= -2$

Using the information provided, draw a neat sketch graph of $y = ax^2 + q$. Hence determine the value of a and q and hence the equation of the graph.

FURTHER QUADRATIC FUNCTIONS

In Grade 10 we discussed quadratic functions of the form $y = ax^2 + q$. We will now focus on quadratic functions of the form $y = a(x+p)^2$ and $y = a(x+p)^2 + q$ where $a \neq 0$.

Graphs of the form $y = a(x+p)^2$

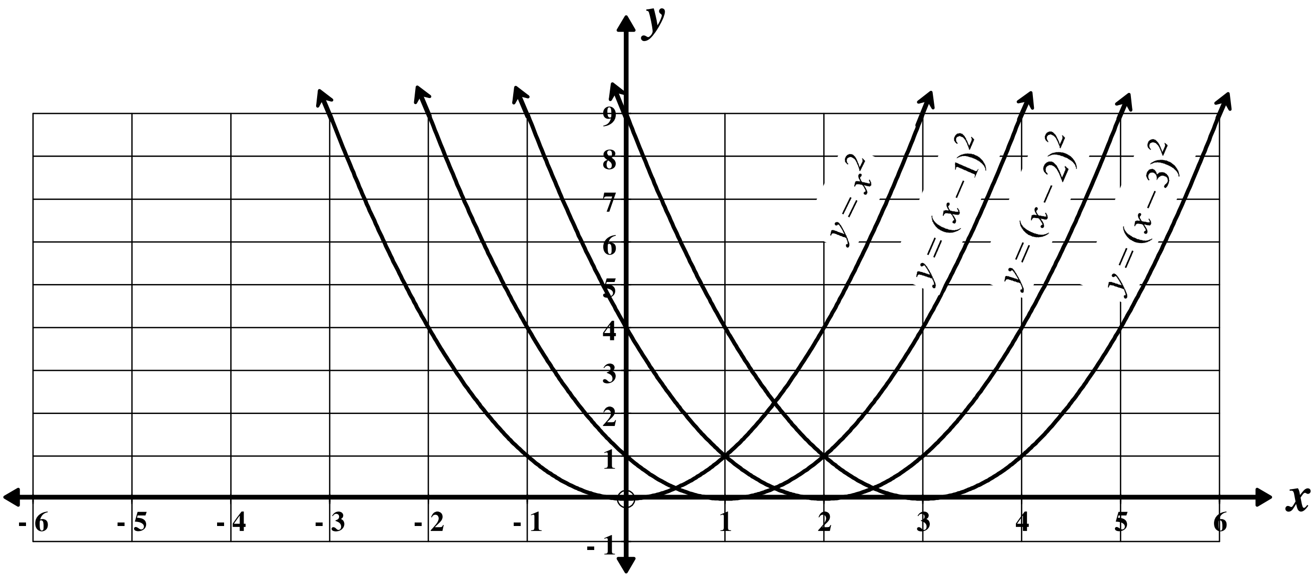
Investigation 1 $y = a(x+p)^2$ where $p < 0$

Consider the following functions:

$y = x^2$, $y = (x-1)^2$, $y = (x-2)^2$, $y = (x-3)^2$

A set of *x*-values $\{-1\,;0\,;1\,;2\,;3\}$ has been selected and the corresponding *y*-values have been calculated for each function. The graphs of the functions have been drawn.

x	-1	0	1	2	3
x^2	1	0	1	4	9
$(x-1)^2$	4	1	0	1	4
$(x-2)^2$	9	4	1	0	1
$(x-3)^2$	16	9	4	1	0



Graph	Turning point of graph	Equation of the axis of symmetry
$y = x^2$	$(0\,;0)$	$x = 0$
$y = (x-1)^2$	$(1\,;0)$	$x = 1$
$y = (x-2)^2$	$(2\,;0)$	$x = 2$
$y = (x-3)^2$	$(3\,;0)$	$x = 3$

Notice:
The equation of the axis of symmetry can be obtained by equating the expression in the brackets to 0 and solving for *x*.

The equation of the axis of symmetry of the graph $y=(x-1)^2$ is:

$x-1=0$

$\therefore x=1$

The equation of the axis of symmetry of the graph $y=(x-2)^2$ is:

$x-2=0$

$\therefore x=2$

The equation of the axis of symmetry of the graph $y=(x-3)^2$ is:

$x-3=0$

$\therefore x=3$

The equation of the axis of symmetry of the graph $y=x^2=(x-0)^2$ is:

$x-0=0$

$\therefore x=0$

Discussion

How do the graphs of $y=(x-1)^2$, $y=(x-2)^2$ and $y=(x-3)^2$ relate to the graph of $y=x^2$?

The graph of $y=(x-1)^2$ is the graph of $y=x^2$ shifted 1 unit to the right.

The graph of $y=(x-2)^2$ is the graph of $y=x^2$ shifted 2 units to the right.

The graph of $y=(x-3)^2$ is the graph of $y=x^2$ shifted 3 units to the right.

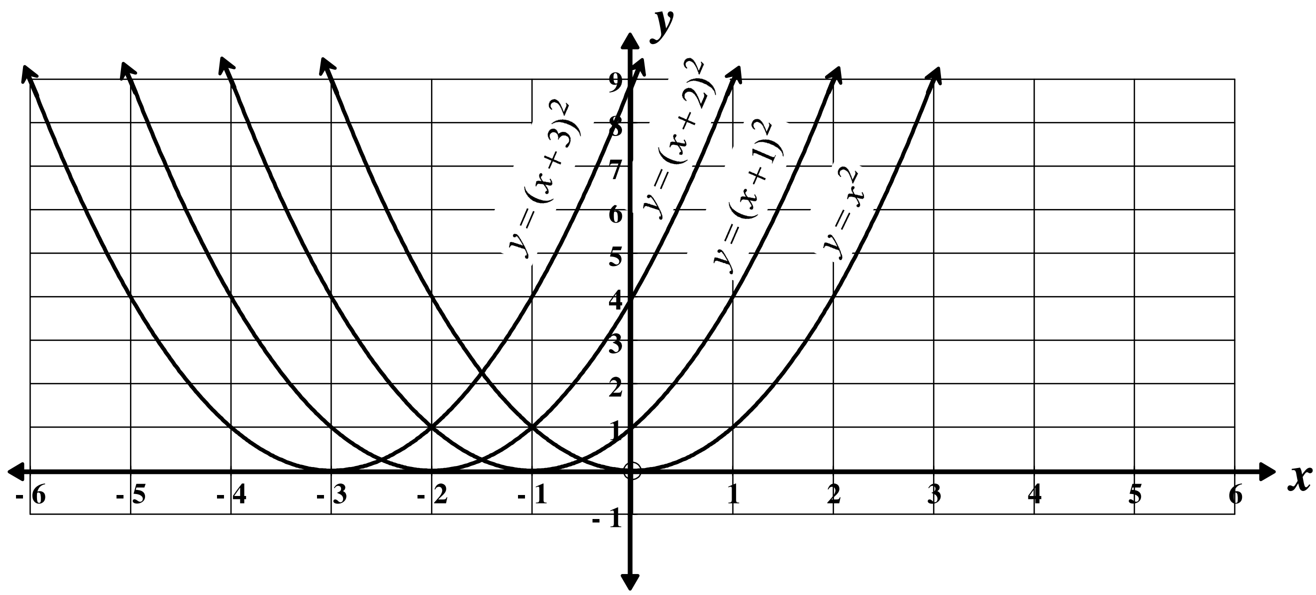
Investigation 2 $y=a(x+p)^2$ where $p>0$

Consider the following functions:

$y=x^2$, $y=(x+1)^2$, $y=(x+2)^2$, $y=(x+3)^2$

A set of *x*-values $\{-3\,;-2\,;-1\,;0\,;1\}$ has been selected and the corresponding *y*-values have been calculated for each function. The graphs of the functions have been drawn.

x	-3	-2	-1	0	1
x^2	9	4	1	0	1
$(x+1)^2$	4	1	0	1	4
$(x+2)^2$	1	0	1	4	9
$(x+3)^2$	0	1	4	9	16



Graph	Turning point of graph	Equation of the axis of symmetry
$y = x^2$	$(0\,;0)$	$x = 0$
$y = (x+1)^2$	$(-1\,;0)$	$x = -1$
$y = (x+2)^2$	$(-2\,;0)$	$x = -2$
$y = (x+3)^2$	$(-3\,;0)$	$x = -3$

Notice:
The equation of the axis of symmetry can be obtained by equating the expression in the brackets to 0 and solving for x.

The equation of the axis of symmetry of the graph $y = (x+1)^2$ is:

$x + 1 = 0$

$\therefore x = -1$

The equation of the axis of symmetry of the graph $y = (x+2)^2$ is:

$x + 2 = 0$

$\therefore x = -2$

The equation of the axis of symmetry of the graph $y = (x+3)^2$ is:

$x + 3 = 0$

$\therefore x = -3$

The equation of the axis of symmetry of the graph $y = x^2 = (x+0)^2$ is:

$x + 0 = 0$

$\therefore x = 0$

Discussion
How do the graphs of $y = (x+1)^2$, $y = (x+2)^2$ and $y = (x+3)^2$ relate to the graph of $y = x^2$?

The graph of $y = (x+1)^2$ is the graph of $y = x^2$ shifted 1 unit to the left.

The graph of $y = (x+2)^2$ is the graph of $y = x^2$ shifted 2 units to the left.

The graph of $y = (x+3)^2$ is the graph of $y = x^2$ shifted 3 units to the left.

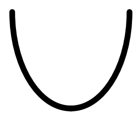
Conclusion:
For graphs of the form $y = a(x+p)^2$:

- The value of a tells us if the graph is concave ($a > 0$) or convex ($a < 0$).
- The equation of the axis of symmetry of the graph $y = a(x+p)^2$ is obtained by putting the expression $x + p = 0$ and solving for x.
- The axis of symmetry passes through the x-coordinate of the turning point of the parabola.
- If $p < 0$, the graph of $y = ax^2$ shifts p units to the right.
- If $p > 0$, the graph of $y = ax^2$ shifts p units to the left.
- The y-intercept of the graph can be determined by putting $x = 0$.
- The x-intercept(s) of the graph can be determined by putting $y = 0$.

EXAMPLE 1

Sketch the graph of the function $f : x \rightarrow 2(x-1)^2$.

Step 1 **(Shape)**



$a = 2 > 0$
$\therefore$ The parabola is concave (happy!)

Step 2 **(Axis of symmetry)**

Put $x - 1 = 0$
$\therefore x = 1$ is the equation of the axis of symmetry.

Step 3 **(Turning point)**

The axis of symmetry passes through the x-coordinate of the turning point.
Therefore, the x-coordinate of the turning point is 1.
Therefore, the coordinates of the turning point are $(1\,;0)$.

Step 4 **(y-intercept)**

Let $x = 0$
$\therefore y = 2(0-1)^2$
$\therefore y = 2$
Therefore the coordinates of the y-intercept are $(0\,;2)$

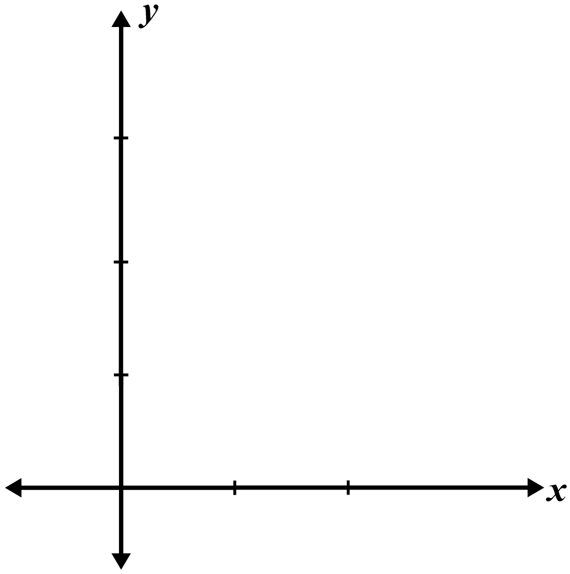
Step 5 **(x-intercept)**

Let $y = 0$
$\therefore 0 = 2(x-1)^2$
$\therefore 0 = (x-1)^2$
$\therefore x = 1$
Therefore the coordinates of the x-intercept are $(1\,;0)$

Step 6 **(Sketch)**

Draw a neat sketch graph of the parabola. Don't forget to plot the point of symmetry as well as the coordinates of the turning point.
(see below for details).

Note: The point $(2\,;2)$ is symmetrical to the point $(0\,;2)$ about the axis of symmetry. We call $(2\,;2)$ a point of symmetry. Don't forget to plot this point when sketching graphs of the form $y = a(x+p)^2$ if it is required.



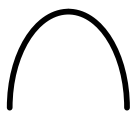
> **Notice**:
> The turning point of a parabola of the form $y = a(x+p)^2$ lies on the x-axis.
> In other words, the coordinates of the turning point and the x-intercept are the same.

<u>EXAMPLE 2</u>

Sketch the graph of the function $f : x \rightarrow -(x+2)^2$.

Step 1 **(Shape)**

$a = -1 < 0$
$\therefore$ The parabola is convex (unhappy!)



Step 2 **(Axis of symmetry)**

Put $x + 2 = 0$
$\therefore x = -2$ is the equation of the axis of symmetry .

Step 3 **(Turning point)**

The axis of symmetry passes through the x-coordinate of the turning point.
Therefore the x-coordinate of the turning point is -2 .
Therefore the coordinates of the turning point are $(-2\,;0)$.

Step 4 **(y-intercept)**

Let $x = 0$

$\therefore y = -(0+2)^2$

$\therefore y = -4$

Therefore the coordinates of the y-intercept are $(0\,;-4)$

Step 5 **(x-intercept)**

Let $y = 0$

$\therefore 0 = -(x+2)^2$

$\therefore (x+2)^2 = 0$

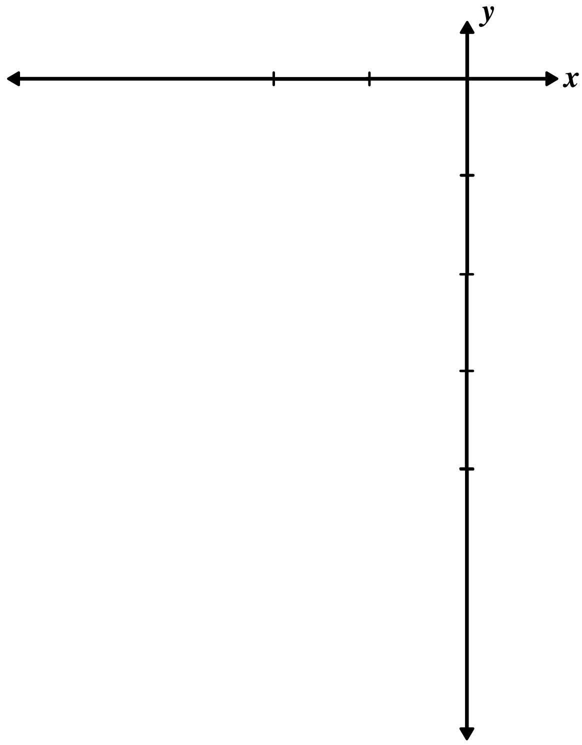
$\therefore x = -2$

Therefore the coordinates of the x-intercept are $(-2\,;0)$

Step 6 **(Sketch)**

Draw the graph

Plot the point of symmetry $(-4\,;-4)$



<u>EXERCISE 2</u>

1. Sketch the graphs of $f(x) = 3x^2$ and $g(x) = 3(x+1)^2$ on the same set of axes. Explain how the graph of $g(x) = 3(x+1)^2$ is formed from the graph of $f(x) = 3x^2$.

2. Sketch the graphs of $f(x) = -2x^2$ and $g(x) = -2(x-1)^2$ on the same set of axes. Explain how the graph of $g(x) = -2(x-1)^2$ is formed from the graph of $f(x) = -2x^2$.

3. Sketch the graph of $y = 2(x+2)^2$. Hence write down the coordinates of the turning point, intercepts with axes, domain and range and the minimum value of the graph.

Graphs of the form $y = a(x+p)^2 + q$

These graphs are the graphs of $y = a(x+p)^2$ shifted q units up or down.

- If $q > 0$, the graph of $y = a(x+p)^2$ shifts q units upwards.
- If $q < 0$, the graph of $y = a(x+p)^2$ shifts q units downwards.

EXAMPLE 3

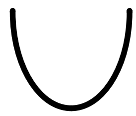
Sketch the graph of $y = (x-2)^2$ and hence the graph of $y = (x-2)^2 + 1$

First consider the function $y = (x-2)^2$

Step 1 **(Shape)**

$a = 1 > 0$

$\therefore$ The parabola is concave (happy!)



Step 2 **(Axis of symmetry)**

Put $x - 2 = 0$

$\therefore x = 2$ is the equation of the axis of symmetry .

Step 3 **(Turning point)**

The axis of symmetry passes through the x-coordinate of the turning point.
Therefore, the coordinates of the turning point are $(2\,;0)$.

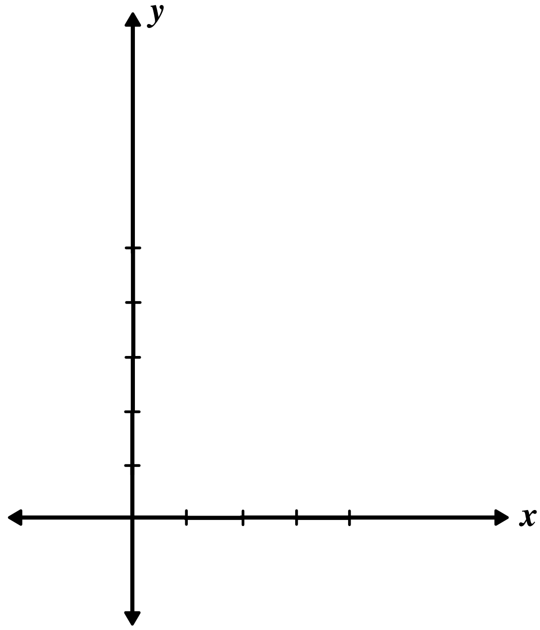
Step 4 **(y-intercept)**

Let $x = 0$

$\therefore y = (0-2)^2$

$\therefore y = 4$

Therefore the coordinates of the y-intercept are $(0\,;4)$



Step 5 **(x-intercept)**

Let $y = 0$

$\therefore 0 = (x-2)^2$

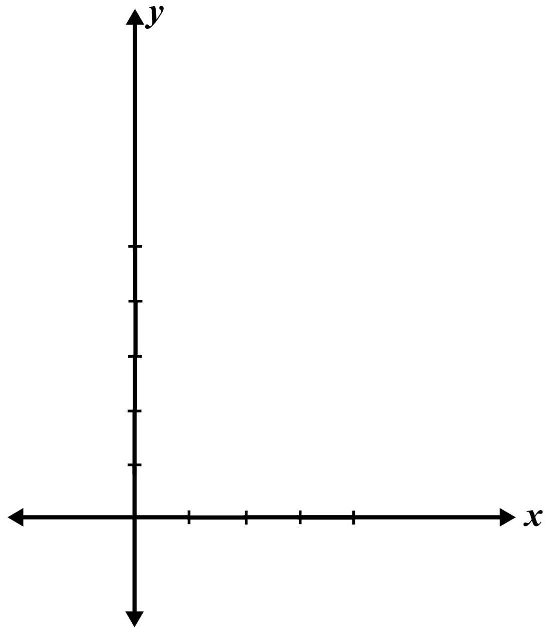
$\therefore x = 2$

Therefore the coordinates of the x-intercept are $(2\,;0)$

Step 6 **(Sketch)**

Draw the graph.

Now sketch the graph of $y = (x-2)^2 + 1$ by shifting the graph of $y = (x-2)^2$ one unit upwards.



Notice: The turning point of the parabola $y=(x-2)^2+1$ is at the point $(2\,;1)$. It is therefore clear that the value of q in the parabola $y=a(x+p)^2+q$ represents the y-coordinate of the turning point.

> **Golden rules for sketching parabolas:**
>
> For graphs of the form $y=a(x+p)^2+q$:
>
> - The value of a tells us if the graph is concave ($a>0$) or convex ($a<0$).
> - The equation of the axis of symmetry of the graph $y=a(x+p)^2+q$ is obtained by putting the expression $x+p=0$ and solving for x.
> - The axis of symmetry passes through the x-coordinate of turning point of the parabola.
> - The graph of $y=a(x+p)^2+q$ is obtained by shifting the graph of $y=ax^2$ by p units to the left or right and then q units up or down.
>   - If $p>0$, the shift is left
>   - If $p<0$, the shift is right
>   - If $q>0$, the shift is upwards
>   - If $q<0$, the shift is downwards
> - The y-coordinate of the turning point is q.
> - The y-intercept of the graph can be determined by putting $x=0$.
> - The x-intercept(s) of the graph can be determined by putting $y=0$.

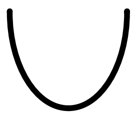
EXAMPLE 4

Sketch the graph of $f(x)=(x-2)^2-4$

Step 1 **(Shape)**

$a=1>0$

$\therefore$ The parabola is concave (happy!)



Step 2 **(Axis of symmetry)**

Put $x-2=0$

$\therefore x=2$ is the equation of the axis of symmetry .

Step 3 **(Turning point)**

The axis of symmetry passes through the x-coordinate of the turning point.

The value of $q=-4$ represents the y-coordinate of the turning point.

Therefore the coordinates of the turning point are $(2\,;-4)$.

Step 4 **(y-intercept)**

Let $x=0$

$\therefore y=(0-2)^2-4$

$\therefore y=0$

Therefore the coordinates of the y-intercept are $(0\,;0)$

Step 5 **(x-intercepts)**

Let $y = 0$

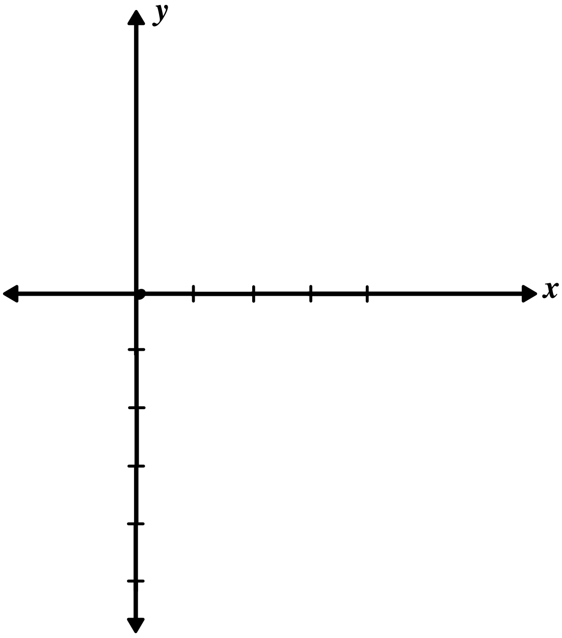
$\therefore 0 = (x-2)^2 - 4$

$\therefore 4 = (x-2)^2$

$\therefore \pm 2 = x - 2$

$\therefore 2 = x - 2$ or $-2 = x - 2$

$\therefore x = 4$ or $x = 0$



Alternatively:

$\therefore 0 = (x-2)^2 - 4$

$\therefore 0 = x^2 - 4x + 4 - 4$

$\therefore 0 = x^2 - 4x$

$\therefore 0 = x(x-4)$

$\therefore x = 0$ or $x = 4$

Step 6 **(Sketch)**

Draw the graph.

Notice that the graph of $y = (x-2)^2 - 4$ is the graph of $y = (x-2)^2$ shifted 4 units downwards.

<u>EXAMPLE 5</u>

(a) Sketch the graph of $f(x) = -x^2$ and $g(x) = -(x+1)^2 - 3$ on the same set of axes.

(b) Explain how the graph of $g(x) = -(x+1)^2 - 3$ relates to the graph of $f(x) = -x^2$.

<u>Solutions</u>

(a) $f(x) = -x^2$

Use a table and then draw this graph on the axes provided below.

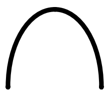
x	-1	0	1
$-x^2$	-1	0	-1

$g(x) = -(x+1)^2 - 3$

Step 1 **(Shape)**

$a = -1 < 0$

$\therefore$ The parabola is convex (unhappy!)



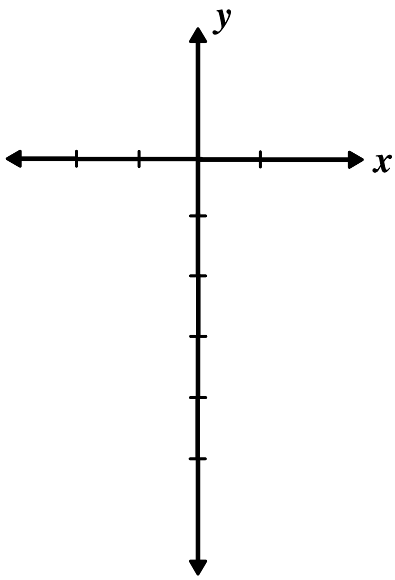
Step 2 **(Axis of symmetry)**

Put $x + 1 = 0$

$\therefore x = -1$ is the equation of the axis of symmetry.

Step 3 **(Turning point)**

The axis of symmetry passes through the x-coordinate of the turning point. Therefore the x-coordinate of the turning point is -1.
The value of $q = -3$ represents the y-coordinate of the turning point. Therefore the coordinates of the turning point are $(-1\,;-3)$.



Step 4 **(y-intercept)**

Let $x = 0$

$\therefore y = -(0+1)^2 - 3$

$\therefore y = -4$

Therefore the coordinates of the y-intercept are $(0\,;-4)$

You can see from the sketch that the graph does not cut the x-axis and therefore has no x-intercepts. So it might be useful to first draw the sketch before attempting to determine the x-intercepts.

Step 5 **(x-intercept)**

Let $y = 0$

$\therefore 0 = -(x+1)^2 - 3$

$\therefore (x+1)^2 = -3$

no real solution

$\therefore$ no x-intercepts

(b) The graph of $g(x) = -(x+1)^2 - 3$ is formed by shifting the graph of $f(x) = -x^2$ one unit to the left and then three units downward.

<u>EXERCISE 3</u>

1. For each function below:

   (1) Draw a neat sketch graph indicating the coordinates of the intercepts with the axes, the coordinates of the turning point and the equation of the axis of symmetry.
   (2) Determine the domain and range.
   (3) Determine the values of x for which the graph increases and decreases.
   (4) Determine the maximum or minimum value of the graph.

(a)	$f(x) = (x+1)^2 - 4$	(b)	$f(x) = -(x+1)^2 + 4$
(c)	$f(x) = (x-1)^2 - 4$	(d)	$f(x) = -(x-1)^2 + 4$
(e)	$f(x) = (x-2)^2 - 1$	(f)	$g(x) = (x+1)^2 + 2$
(g)	$g(x) = -(x+1)^2 - 2$	(h)	$g(x) = (x-2)^2 - 9$
(i)	$g(x) = 2(x+1)^2 - 2$	(j)	$g(x) = (x-4)^2 - 2$

2. (a) Sketch the graph of $f(x) = 2x^2$ and $g(x) = 2(x-2)^2 + 1$ on the same set of axes.

(b) Explain how the graph of $g(x) = 2(x-2)^2 + 1$ relates to the graph of $f(x) = 2x^2$.

3. Consider $f(x) = x^2 - 4$. Determine the equation of the new graph formed if the graph of $f(x) = x^2 - 4$ is:

(a) shifted 2 units to the left.
(b) shifted 2 units to the right.
(c) shifted 4 units upwards.
(d) shifted 1 unit downwards.
(e) shifted 2 units to the left and 3 units upwards.
(f) shifted 2 units to the right and 2 units downwards.
(g) reflected about the x-axis.

4. (a) Sketch the graph of $f(x) = x^2$ on a set of axes.

(b) Now draw neat sketch graphs of the following on different sets of axes:

(1)	$y = 2f(x)$	(2)	$y = f(2x)$
(3)	$y = f(x) + 2$	(4)	$y = f(x-2)$
(5)	$y = f(x+2)$	(6)	$y = f(x-1) + 2$
(7)	$y = -f(x)$	(8)	$y = -f(x) - 2$

(c) Now explain how each of the graphs in (b) relate to the graph of $f(x) = x^2$.

Graphs of the form $y = ax^2 + bx + c$

Graphs of the form $y = ax^2 + bx + c$ can be converted into the form $y = a(x+p)^2 + q$ by **completing the square**.

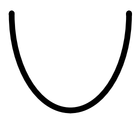
EXAMPLE 6

Sketch the graph of $f(x) = x^2 - 2x - 8$

Step 1 **(Shape)**

$a = 1 > 0$

$\therefore$ The parabola is concave (happy!)



In order to get the **axis of symmetry and turning point**, we will make use of the technique of **completing the square** to rewrite the equation $y = x^2 - 2x - 8$ in the form $y = a(x+p)^2 + q$.

Add and subtract the square of half the coefficient of x:

$y = x^2 - 2x - 8$

$\therefore y = x^2 - 2x + \left[\frac{1}{2}(-2)\right]^2 - \left[\frac{1}{2}(-2)\right]^2 - 8$

$\therefore y = x^2 - 2x + (-1)^2 - (-1)^2 - 8$

$\therefore y = x^2 - 2x + 1 - 1 - 8$

$\therefore y = x^2 - 2x + 1 - 9$

$\therefore y = (x-1)^2 - 9$

Now we can determine the equation of the axis of symmetry and turning point.

Step 2 **(Axis of symmetry)**

Put $x - 1 = 0$
$\therefore x = 1$

Step 3 **(Turning point)**

The axis of symmetry passes through the x-coordinate of the turning point.
Therefore the x-value of the turning point is 1.
The value of $q = -9$ represents the y-coordinate of the turning point.
Therefore the coordinates of the turning point are $(1\,;-9)$.

Step 4 **(y-intercept)**

Use the original equation to get the y-intercept.
Let $x = 0$

$\therefore y = (0)^2 - 2(0) - 8$

$\therefore y = -8$

Therefore the coordinates of the y-intercept are $(0\,;-8)$.

Notice that the constant term of $y = x^2 - 2x - 8$ represents the y-intercept.
Actually, the y-intercept of any quadratic function $y = ax^2 + bx + c$ is always the value of c.

Step 5 **(x-intercepts)**

Using the original form:

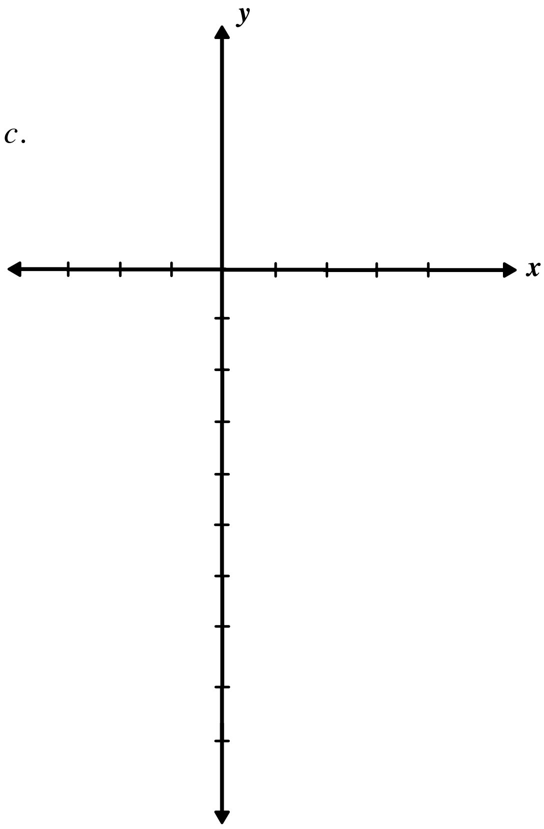
$0 = x^2 - 2x - 8$

$\therefore 0 = (x-4)(x+2)$

$\therefore x = 4 \quad \text{or} \quad x = -2$

Step 6 **(Sketch)**

Draw the graph

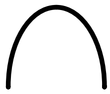


EXAMPLE 7

Sketch the graph of $f(x) = -x^2 - 4x$

Step 1 **(Shape)**

$a = -1 < 0$
$\therefore$ The parabola is convex (unhappy!)



In order to get the axis of symmetry and turning point, we will make use of the technique of completing the square to rewrite the equation $y = -x^2 - 4x$ in the form $y = a(x + p)^2 + q$.

First take out the negative and then add and subtract the square of half the coefficient of x:

$y = -x^2 - 4x$

$\therefore y = -(x^2 + 4x)$

$\therefore y = -\left[x^2 + 4x + \left(\frac{1}{2}(4)\right)^2 - \left(\frac{1}{2}(4)\right)^2 \right]$

$\therefore y = -\left[x^2 + 4x + 4 - 4 \right]$

$\therefore y = -\left[(x+2)^2 - 4 \right]$

$\therefore y = -(x+2)^2 + 4$

Now we can determine the equation of the axis of symmetry and turning point.

Step 2 **(Axis of symmetry)**

Put $x + 2 = 0$
$\therefore x = -2$

Step 3 **(Turning point)**

The axis of symmetry passes through the x-coordinate of the turning point.
Therefore the x-value of the turning point is -2 .
The value of $q = 4$ represents the y-coordinate of the turning point.
Therefore the coordinates of the turning point are $(-2\,;4)$.

Step 4 **(y-intercept)**

Let $x = 0$ in the original.
$\therefore y = -(0)^2 - 4(0) = 0$
Therefore the coordinates of the y-intercept are $(0\,;0)$.

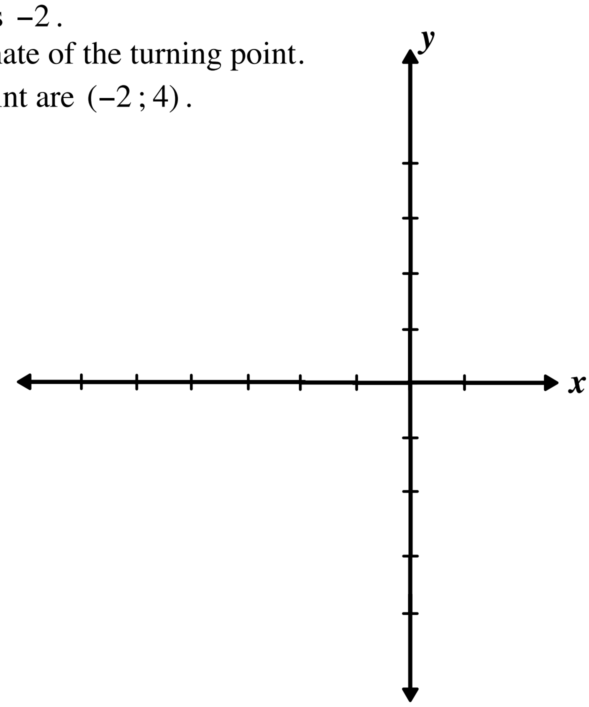
Step 5 **(x-intercepts)**
Let $y = 0$

$0 = -x^2 - 4x$

$\therefore x^2 + 4x = 0$

$\therefore x(x+4) = 0$

$\therefore x = 0$ or $x = -4$



There is another method of determining the x-coordinate of the turning point of a parabola with equation in the form $y = ax^2 + bx + c$. By completing the square we obtain the following:

$$y = ax^2 + bx + c$$

$$\therefore y = a\left[x^2 + \frac{b}{a}x\right] + \frac{c}{a}$$

$$\therefore y = a\left[x^2 + \frac{b}{a}x + \left(\frac{1}{2}\left(\frac{b}{a}\right)\right)^2 - \left(\frac{1}{2}\left(\frac{b}{a}\right)\right)^2\right] + \frac{c}{a}$$

$$\therefore y = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + \frac{c}{a}$$

$$\therefore y = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right] + \frac{c}{a}$$

$$\therefore y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + \frac{c}{a}$$

$$\therefore y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \frac{c}{a}$$

$$\therefore y = a\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a}$$

$$\therefore y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Clearly, it can be deduced that the x-coordinate of the turning point is given by the formula: $\boldsymbol{x = -\frac{b}{2a}}$

To get the y-coordinate of the turning point, simply substitute the x-value obtained into the original equation.

EXAMPLE 8

Determine the coordinates of the turning point of $y = x^2 - 2x - 8$ using the above method.

$a = 1$ and $b = -2$

$$x = -\frac{b}{2a}$$

$$\therefore x = -\frac{(-2)}{2(1)} = 1$$

Substitute back into original to get y:

$$y = (1)^2 - 2(1) - 8$$

$$\therefore y = 1 - 2 - 8$$

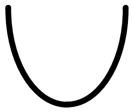
$$\therefore y = -9$$

The coordinates of the turning point are therefore $(1\,;-9)$

EXAMPLE 9

Sketch the graph of $f(x) = x^2 - 4x + 3$

Step 1 **(Shape)**



$a = 1 > 0$

$\therefore$ The parabola is concave (happy!)

Now we can determine the equation of the axis of symmetry and the turning point using one of the two methods discussed:

- Completing the square, or
- Using the formula $x = -\dfrac{b}{2a}$

Step 2 **(Axis of symmetry)**

Method 1

$y = x^2 - 4x + 3$

$\therefore y = x^2 - 4x + \left[\frac{1}{2}(-4)\right]^2 - \left[\frac{1}{2}(-4)\right]^2 + 3$

$\therefore y = x^2 - 4x + 4 - 4 + 3$

$\therefore y = (x-2)^2 - 1$

Put $x - 2 = 0$

$\therefore x = 2$

Method 2

$y = x^2 - 4x + 3$

$a = 1 \quad b = -4$

$\therefore x = -\dfrac{(-4)}{2(1)}$

$\therefore x = 2$

Step 3 **(Turning point)**

Method 1

The axis of symmetry passes through the x-coordinate of the turning point.

Therefore the x-value of the turning point is 2.

The value of $q = -1$ represents the y-coordinate of the turning point.

Therefore the coordinates of the turning point are $(2\,;-1)$.

Method 2

Substitute back into original to get y:

$y = (2)^2 - 4(2) + 3$

$\therefore y = 4 - 8 + 3$

$\therefore y = -1$

The turning point is therefore $(2\,;-1)$

Step 4 **(*y*-intercept)**

The coordinates of the y-intercept are $(0\,;3)$.

Step 5 **(*x*-intercepts)**

Let $y = 0$

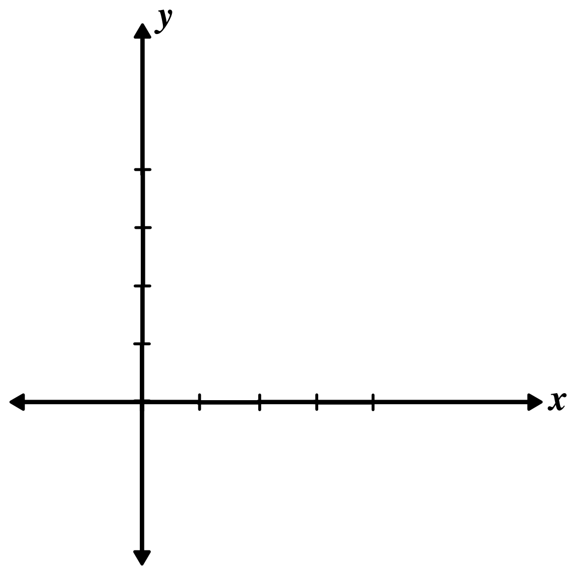
$0 = x^2 - 4x + 3$

$\therefore 0 = (x-3)(x-1)$

$\therefore x = 3 \quad \text{or} \quad x = 1$

Step 6 **(Sketch)**

Draw the graph.



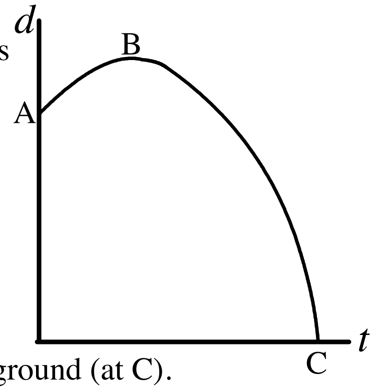
> Notice:
> The x-coordinate of the turning point can also be determined as follows:
>
> $x_{TP} = \dfrac{\text{sum of } x\text{-intercepts}}{2}$
>
> $x_{TP} = \dfrac{1+3}{2} = 2$

<u>EXERCISE 4</u>

1. For each quadratic function below, sketch the graph showing the axis of symmetry and all critical points (intercepts with the axes, turning point). Hence determine the domain and range for each function. Also write down the maximum or minimum value of each function. State the values of x for which the function increases and decreases.

(a)	$f(x) = x^2 - 6x + 5$	(b)	$g(x) = 8 - 2x - x^2$
(c)	$y = 2x^2 + 8x - 10$	(d)	$h(x) = -x^2 + 6x$
(e)	$f : x \rightarrow 2(x+3)^2 - 8$	(f)	$f(x) = -(x+2)^2 + 1$
(g)	$y = -x^2 + 4x + 12$	(h)	$y = x^2 - 4x + 4$
(i)	$y = x^2 - 9$	(j)	$p(x) = 25 - x^2$
(k)	$y = x^2 + 2$	(l)	$y = 8 - 2x^2$
(m)	$f(x) = -x^2 - 2$	(n)	$y = -4x^2$
(o)	$g(x) = \frac{1}{2}x^2$	(p)	$y = -(x+2)^2 + 6$
(q)	$m(x) = x^2 + x + 1$	(r)	$f(x) = -2(x-1)(x+1)$
(s)	$g(x) = \left(x + \frac{3}{2}\right)^2 - \frac{21}{4}$	(t)	$f : x \rightarrow \frac{1}{2}x^2 + 2x + 2$

2. A small handgun is fired from the window of a building. The pathway of the bullet is described by the equation $d = -\frac{1}{10}t^2 + 2t + 30$ where t represents time in seconds and d represents the height of the bullet above the ground in metres. The graph of the pathway of the ball is shown in the diagram alongside.



(a) Calculate the height of the bullet above the ground before the gun is fired (at A).
(b) Calculate the maximum height that the bullet reaches above the ground (at B).
(c) Calculate the time taken for the bullet to hit the ground (at C).

FURTHER HYPERBOLIC FUNCTIONS

In Grade 10 we discussed hyperbolic functions of the form $y = \frac{a}{x} + q$. We will now focus on hyperbolic functions of the form $y = \frac{a}{x+p} + q$ where $x + p \neq 0$.

Consider $f(x) = \frac{2}{x-2} + 1$.

It is clear from the revision of Grade 10 hyperbolas that the graph of $y = \frac{2}{x-2} + 1$ has a **horizontal asymptote** at $y = 1$.

It is also the case that $x - 2 \neq 0$, i.e. $x \neq 2$ because then the denominator will be zero and the expression $\frac{2}{x-2}$ would be undefined.

In other words, the graph is not defined for $x = 2$. The graph, therefore, has a **vertical asymptote** at $x = 2$.

The shift for these graphs works the same as for the parabola. The "mother" graph $y = \frac{2}{x}$ shifts 2 units to the right and then 1 unit upwards to form the graph of $y = \frac{2}{x-2} + 1$.

Golden rules for sketching hyperbolas of the form $y = \frac{a}{x+p} + q$

1. Determine the **shape**:

(The dotted lines are the asymptotes)

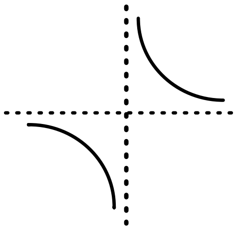
2. Write down the **asymptotes** and draw them on a set of axes:
   Vertical asymptote: $x + p = 0$
   Horizontal asymptote: $y = q$
3. Plot the **four "mother" graph points** on your set of axes.
4. Shift the **four "mother" graph points** left or right, up or down.
5. Determine the y-intercept: let $x = 0$.
6. Determine the x-intercepts: let $y = 0$.
7. Draw the newly formed graph.

EXAMPLE 10

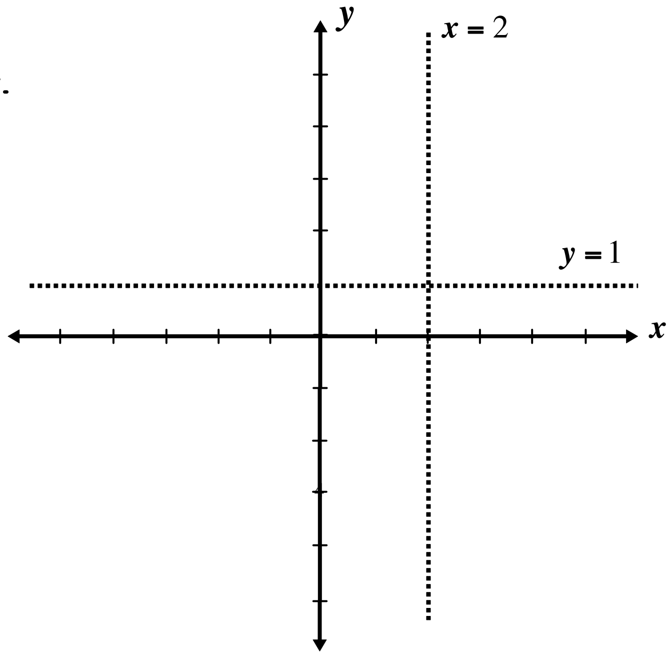
Sketch the graph of $f(x) = \dfrac{2}{x-2} + 1$

Step 1 **Shape**

$a = 2 > 0$



Step 2 **Asymptotes**

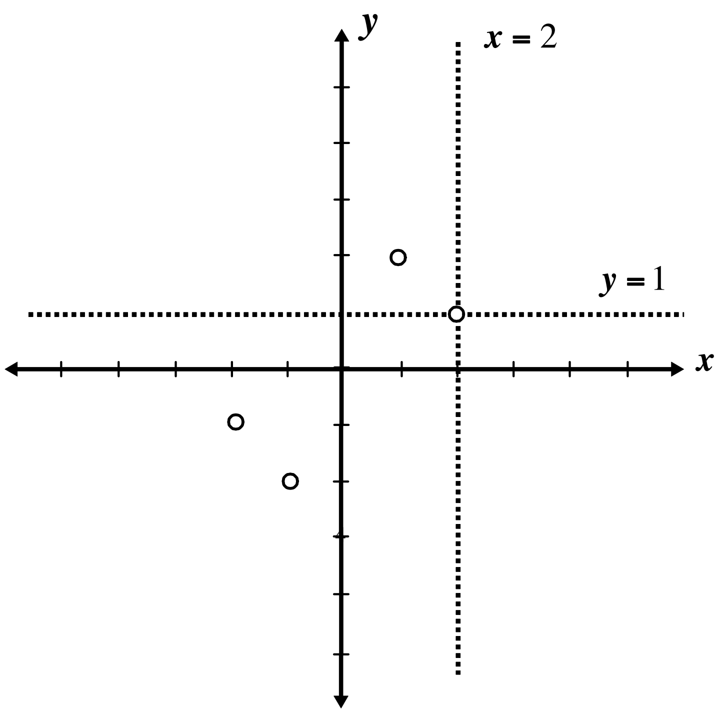


Vertical asymptote:
$x - 2 = 0$
$\therefore x = 2$

Horizontal asymptote:
$y = 1$

Step 3 **(Four "mother" graph points)**

x	-2	-1	1	2
$\dfrac{2}{x}$	-1	-2	2	1

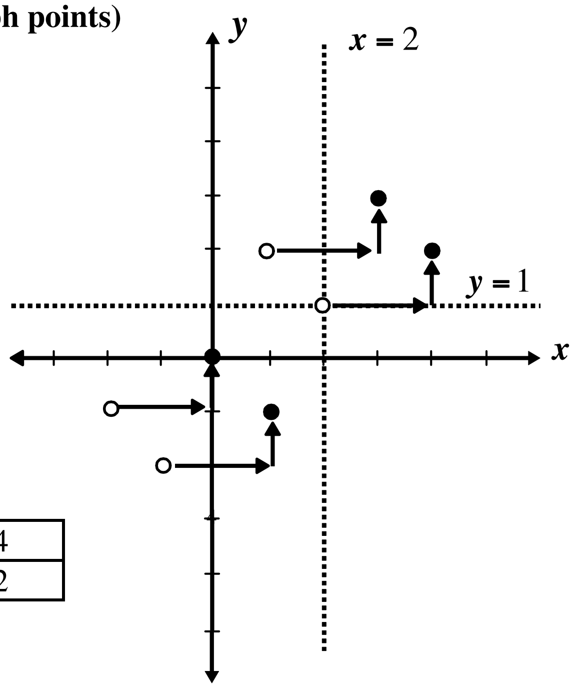


Step 4 **(Shifting of "mother" graph points)**

Vertical asymptote of "mother" graph has shifted 2 units right. The horizontal asymptote of "mother" graph has shifted 1 unit up. Therefore shift the "mother" graph points 2 units right and 1 unit up.

Using the table on the previous page, add 2 to each x-value and add 1 to each y-value. Plot the new points.

new x	0	1	3	4
new y	0	-1	3	2



Step 5 **(y-intercept)**

$$y = \frac{2}{0-2} + 1$$

$$\therefore y = 0$$

Step 6 **(x-intercept)**

$$0 = \frac{2}{x-2} + 1$$

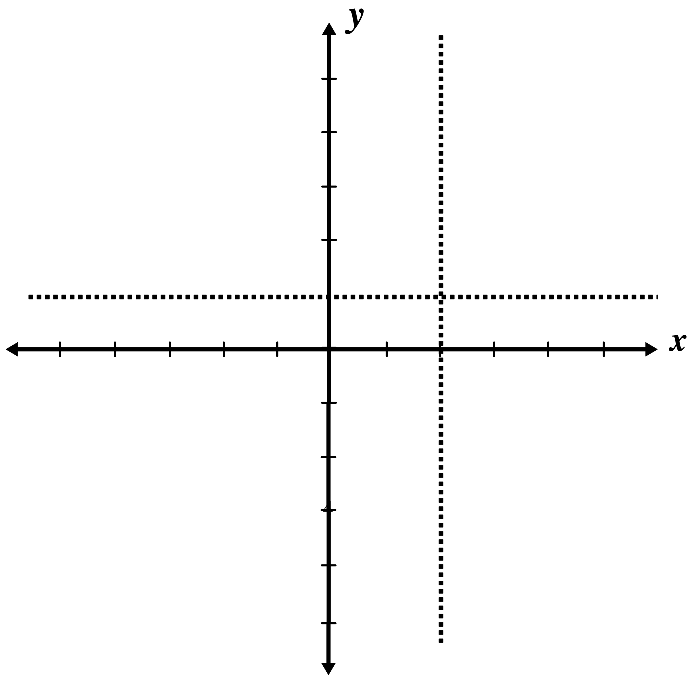
$$\therefore 0 = 2 + 1(x-2)$$

$$\therefore 0 = 2 + x - 2$$

$$\therefore x = 0$$

Step 7

Draw the newly formed graph.

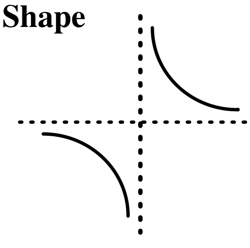


<u>EXAMPLE 11</u>

Sketch the graph of $f(x) = \dfrac{3}{x+1} - 2$

Step 1 **Shape**

$a = 3 > 0$



Step 2 **Asymptotes**

Vertical asymptote: $x+1=0$

$\therefore x=-1$

Horizontal asymptote: $y=-2$

Step 3 **(Four "mother" graph points)**

x	-3	-1	1	3
$\frac{3}{x}$	-1	-3	3	1

Step 4 **(Shifting of "mother" graph points)**

Vertical asymptote of "mother" graph has shifted 1 unit left.
Horizontal asymptote of "mother" graph has shifted 2 units down.
Therefore shift the "mother" graph points 1 unit left and 2 units down.
Using the table above, subtract 1 from each x-value and subtract 2 from each y-value. Plot the new points.

new x	-4	-2	0	2
new y	-3	-5	1	-1

Step 5 **(y-intercept)**

$y=\frac{3}{0+1}-2$

$\therefore y=1$

Step 6 **(x-intercept)**

$0=\frac{3}{x+1}-2$

$\therefore 0=3-2(x+1)$

$\therefore 0=3-2x-2$

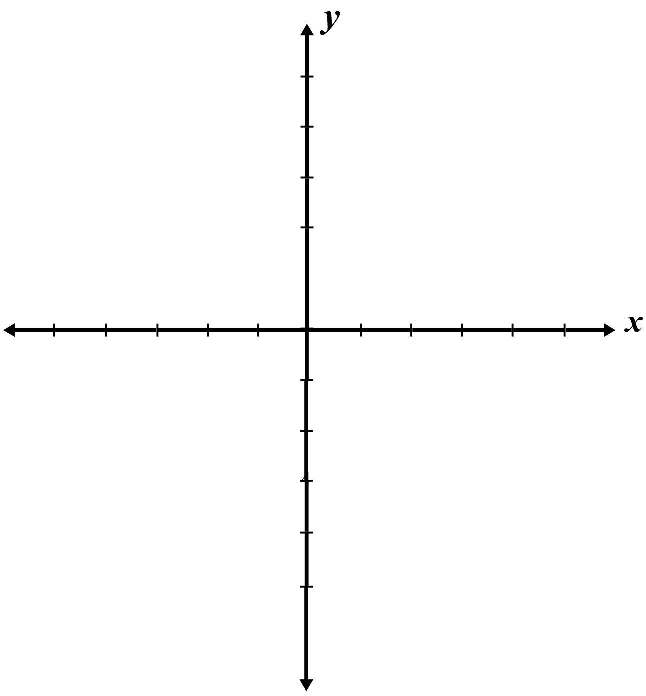
$\therefore 0=1-2x$

$\therefore 2x=1$

$\therefore x=\frac{1}{2}$

Step 7

Draw the newly formed graph.



EXERCISE 5

1. For each hyperbolic function below, sketch the graph showing the asymptotes and the intercepts with the axes. Hence determine the domain and range for each function. State the values of x for which the function increases or decreases.

(a) $f(x)=\dfrac{2}{x+1}-1$ (b) $g(x)=\dfrac{-2}{x+1}+1$

(c) $y=\dfrac{2}{x-1}+1$ (d) $f(x)=-\dfrac{2}{x-1}+1$

(e) $p(x)=\dfrac{3}{x+2}-3$ (f) $f(x)=\dfrac{-3}{x+2}+3$

(g) $g(x)=\dfrac{4}{x-3}+2$ (h) $f:x\to\dfrac{4}{x-3}-2$

(i) $y=\dfrac{1}{x-1}+2$ (j) $g(x)=\dfrac{-1}{x-1}-2$

(k) $f(x)=\dfrac{2}{x-3}$ (l) $f(x)=\dfrac{2}{x+3}$

2. Consider the graph of $f(x)=\dfrac{2}{x}$. Determine the equation of the new graph formed if the graph of $f(x)=\dfrac{2}{x}$ is:

(a) shifted 2 units upwards. (b) shifted 2 units downwards.
(c) shifted 2 units to the left. (d) shifted 2 units to the right.
(e) shifted 2 units to the left and then 3 units upwards.
(f) shifted 2 units to the right and then 3 units downwards.
(g) reflected about the y-axis.
(h) reflected about the x-axis.

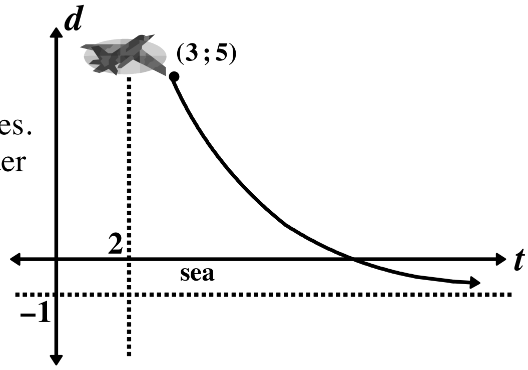
3. (a) Sketch the graph of $f(x)=\dfrac{3}{x}$ on a set of axes.

(b) Now draw neat sketch graphs of the following on the different sets of axes:

(1) $y=2f(x)$ (2) $y=f(x)+2$
(3) $y=f(x-2)$ (4) $y=f(x+2)$
(5) $y=f(x-1)+2$ (6) $y=f(-x)$

(c) Now explain how each of the graphs in (b) relate to the graph of f.

4. The movement of a guided missile fired from an aeroplane is represented by the following hyperbolic graph where t represents time in seconds and d the displacement of the missile from the surface of the sea in kilometres. The missile is fired three seconds after the order is given to fire. The order is given at $t=0$. The missile is fired at a displacement of 5 km above the surface of the sea. The equation of the displacement of the missile is given by $d=\dfrac{a}{t-2}-1$



(a) How long will it take for the missile to be 1km above the surface of the sea?
(b) After how many seconds will the missile hit the surface of the sea?
(c) Explain why the missile will never be displaced more than 1 km below the surface of the sea?

FURTHER EXPONENTIAL FUNCTIONS

In Grade 10, we discussed exponential functions of the form $y = ab^x + q$ where $b > 0$; $b \neq 1$.

Discussion

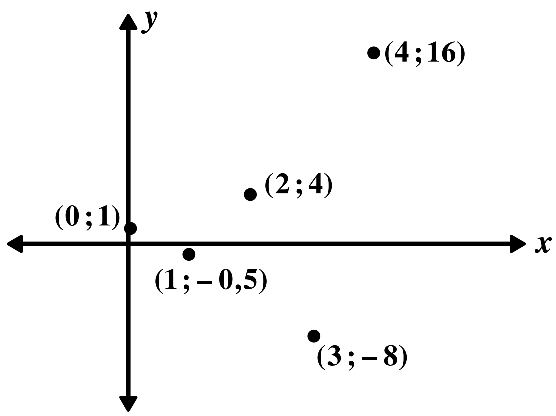
Why do exponential functions have the restrictions $b > 0$ and $b \neq 1$?

(a) Let's consider the case where $b = 1$

The graph of $y = (1)^x$, for example, is a horizontal line $y = 1$ and not an exponential function.

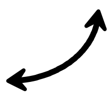
(b) Let's consider the case where $b < 0$

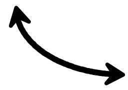
The graph of $y = (-2)^x$, for example, is not an exponential function.



Remember the two basic shapes of the exponential graphs:

If $b > 1$, the shape is increasing If $0 < b < 1$, the shape is decreasing.





The graphs increase or decrease for all values of *x*.

We will now focus on exponential functions of the form $y = ab^{x+p}$ and $y = ab^{x+p} + q$ where $b > 0$ and $b \neq 1$.

Exponential graphs of the form $y = ab^{x+p}$

EXAMPLE 12

Consider the functions $f(x) = 2^x$ $g(x) = 2^{x-1}$ $h(x) = 2^{x+1}$

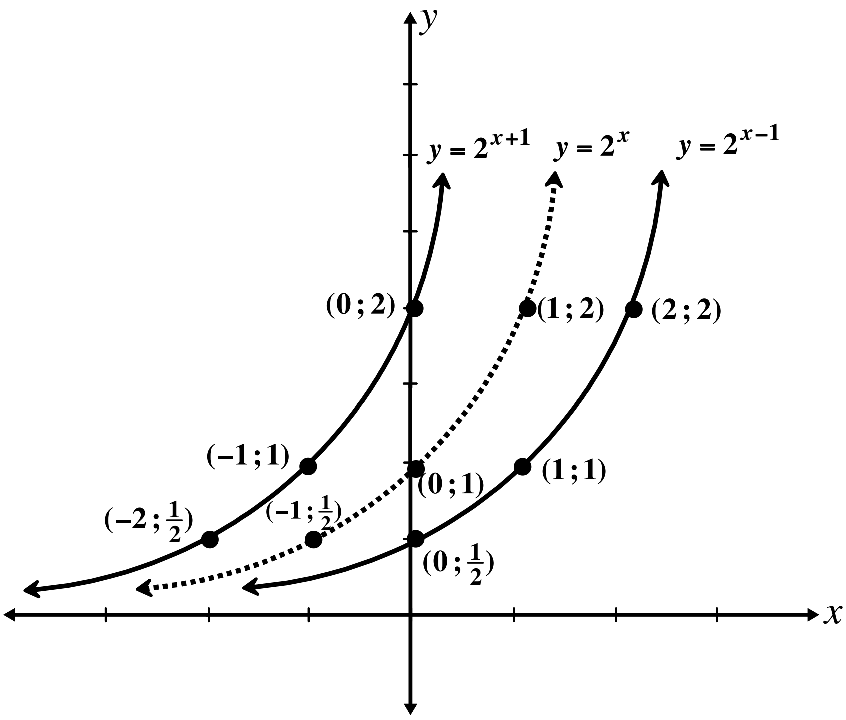
A set of *x*-values has been selected in each case and the corresponding *y*-values have been calculated for each function.

The tables are given as is to show you how the graphs of $g(x) = 2^{x-1}$ and $h(x) = 2^{x+1}$ relate to the graph of $f(x) = 2^x$.

x	-1	0	1
2^x	$\frac{1}{2}$	1	2

x	0	1	2
2^{x-1}	$\frac{1}{2}$	1	2

x	-2	-1	0
2^{x+1}	$\frac{1}{2}$	1	2



Discussion

(a) How does the graph of $g(x) = 2^{x-1}$ relate to the graph of $f(x) = 2^x$?

The graph of $y = 2^{x-1}$ is the graph of $y = 2^x$ shifted 1 unit to the right.

(b) How does the graph of $h(x) = 2^{x+1}$ relate to the graph of $f(x) = 2^x$?

The graph of $y = 2^{x+1}$ is the graph of $y = 2^x$ shifted 1 unit to the left.

Conclusion

Consider the function $y = ab^{x+p}$.

If $p < 0$, the graph of $y = ab^{x+p}$ is the graph of $y = ab^x$ shifted p units to the right.

If $p > 0$, the graph of $y = ab^{x+p}$ is the graph of $y = ab^x$ shifted p units to the left.

Exponential graphs of the form $y = ab^{x+p} + q$

The graph of $y = ab^{x+p} + q$ is formed by shifting the graph of the function $y = ab^x$ as follows:
p units horizontally (left or right) and then q units vertically (up or down).

EXAMPLE 13

(a) Sketch the graphs of $f(x) = 2^x$ and $g(x) = 2^{x+1} + 1$ on the same set of axes.

$f(x) = 2^x$

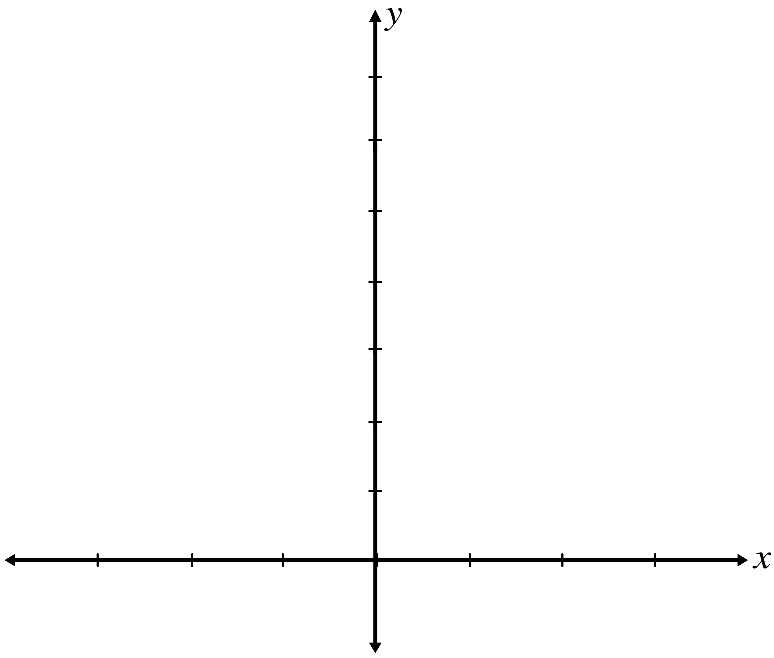
x	-1	0	1
2^x	$\frac{1}{2}$	1	2

$g(x) = 2^{x+1} + 1$

Shift x-values 1 unit left and add 1 unit to y-values

new x	-2	-1	0
new y	$1\frac{1}{2}$	2	3

Horizontal asymptote: $y = 1$

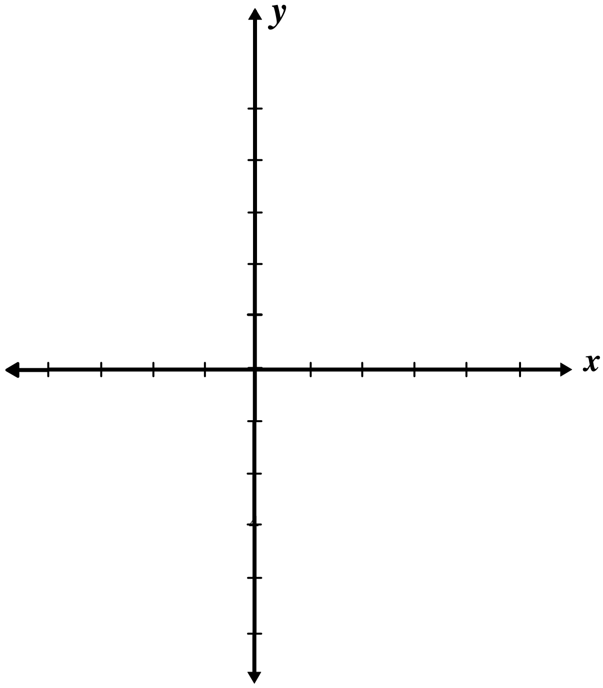


(b) How does the graph of $g(x) = 2^{x+1} + 1$ relate to the graph of $f(x) = 2^x$?

The graph of $y = 2^{x+1} + 1$ is the graph of $y = 2^x$ shifted 1 unit left and 1 unit upwards.

EXAMPLE 14

Sketch the graph $f(x) = 2.2^{x-1} - 4$



Horizontal asymptote is $y = -4$

$f(x) = 2.2^{x}$ (Note: $2.2^{x} \neq 4^{x}$)

x	-1	0	1
2.2^{x}	1	2	4

$g(x) = 2^{x-1} - 4$

Shift x-values 1 unit right and subtract 4 units from the y-values

new *x*	0	1	2
new *y*	-3	-2	0

Note:
You may also use $x \in \{-1\,;0\,;1\}$ as your selected x-values. The points will still lie on the graph of $y = f(x) = 2.2^{x-1} - 4$

x	-1	0	1
$2.2^{x-1} - 4$	$-3\frac{1}{2}$	-3	-2

The x-intercept can be determined as follows:

Let $y = 0$

$0 = 2.2^{x-1} - 4$

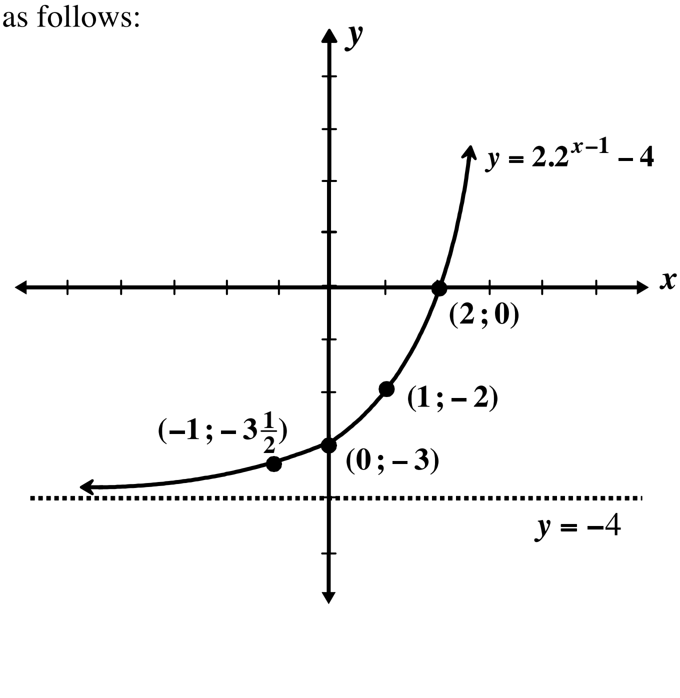
$\therefore -2.2^{x-1} = -4$

$\therefore 2^{x-1} = 2$

$\therefore 2^{x-1} = 2^{1}$

$\therefore x - 1 = 1$

$\therefore x = 2$



EXERCISE 6

1. For each exponential function below, sketch the graph showing the asymptotes and any intercepts with the axes. Hence determine the domain and range for each function. State the values of x for which the function increases or decreases.

(a) $f(x) = 2^{x+2}$ (b) $g(x) = 2^{x-2}$

(c) $y = 2^{x+2} - 1$ (d) $p(x) = 2^{x-2} + 1$

(e) $y = -2^{x+1}$ (f) $y = -2^{x+1} + 2$

(g) $f(x) = 2.3^{x-1} + 1$ (h) $y = \frac{1}{2}(2)^{x-1} - 1$

(i) $y = \left(\frac{1}{2}\right)^{x+1} - 1$ (j) $y = \left(\frac{1}{2}\right)^{x-1} + 1$

2. (a) Sketch the graph of $f(x) = 2^x$ on a set of axes.

(b) Now draw neat sketch graphs of the following on the different sets of axes:

(1) $y = 2f(x)$ (2) $y = f(x) + 2$

(3) $y = f(x-2)$ (4) $y = f(x+2)$

(5) $y = f(x-1) + 2$ (6) $y = -f(x)$

(7) $y = f(-x)$ *(8) $y = f(-x+1)$ (Be careful!)

(c) Now explain how each of the graphs in (b) relate to the graph of $f(x) = 2^x$.

(d) Determine the equation of the reflection of $f(x) = 2^x$ about the x-axis.

(e) Determine the equation of the reflection of $f(x) = 2^x$ about the y-axis.

(f) Determine the equation of the new graph formed if the graph of $f(x) = 2^x$ is shifted 3 units to the right.

(g) Determine the equation of the new graph formed if the graph of $f(x) = 2^x$ is shifted 3 units to the left.

(h) Determine the equation of the new graph formed if the graph of $f(x) = 2^x$ is shifted 3 units upwards.

(i) Determine the equation of the new graph formed if the graph of $f(x) = 2^x$ is shifted 3 units downwards.

(j) Determine the equation of the new graph formed if the graph of $f(x) = 2^x$ is shifted 1 unit to the right and then 2 units upwards.

3. A water plant grows on the surface of a dam. The surface area covered by the plant doubles every day. At the beginning of a certain day, the plant covered one square metre of the surface.

(a) Complete the following table:

Time n (in days)	0	1	2	3	4	5	6	n
Area A covered (square metres)	1	2	4	8	16			

(b) Sketch the graph of the area A against time n.
(c) What is the equation of this graph?
(d) What will the area be on the 9^{th} day?
(e) Determine the day on which the area reached 1024 square meters.
(f) If the table changes as follows, determine the equation of the new graph. Explain what has happened graphically.

Time n (in days)	0	1	2	3	4	5	6	n
Area A covered (square metres)	2	3	5	9				

DETERMINING THE EQUATION OF A PARABOLA

TYPE 1

Here we are given the x-intercepts and one point on the graph. We will use the structure $y = a(x - x_1)(x - x_2)$ where x_1 and x_2 are the x-intercepts.

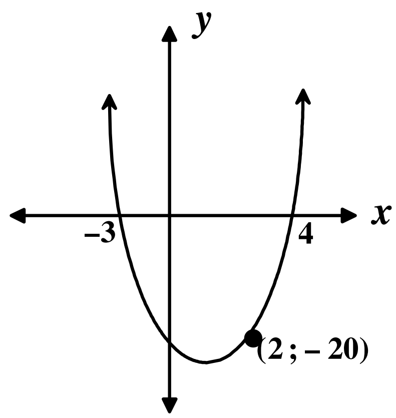
EXAMPLE 15

Determine the equation of the parabola in the form $f(x) = ax^2 + bx + c$.

$y = a(x - x_1)(x - x_2)$

$\therefore y = a(x - (-3))(x - 4)$

$\therefore y = a(x + 3)(x - 4)$



Substitute $(2\,;-20)$ to find the value of a.

$-20 = a(2 + 3)(2 - 4)$

$\therefore -20 = a(5)(-2)$

$\therefore -20 = -10a$

$\therefore a = 2$

$\therefore y = 2(x + 3)(x - 4)$

$\therefore y = 2(x^2 - x - 12)$

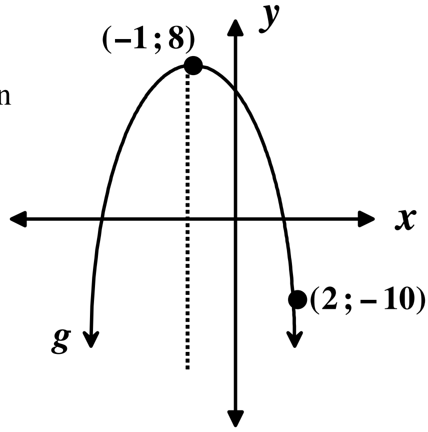
$\therefore f(x) = 2x^2 - 2x - 24$

TYPE 2

Here we are given the turning point and one other point on the graph. We make use of the structure $y = a(x + p)^2 + q$

EXAMPLE 16

Determine the equation of g in the form $y = ax^2 + bx + c$



The axis of symmetry is given by $x = -1$.
$\therefore x + 1 = 0$
From the work on parabolas, it is clear that the expression $x + 1$ is in the brackets of the equation for a parabola.
Also, the value of q is 8 (the y-value of the turning point).
$\therefore y = a(x+1)^2 + 8$
Now substitute the point $(2 ; -10)$ which lies on the graph of the parabola.

$-10 = a(2+1)^2 + 8$

$\therefore -18 = a(3)^2$

$\therefore -18 = 9a$

$\therefore a = -2$

$\therefore y = -2(x+1)^2 + 8$

$\therefore y = -2(x^2 + 2x + 1) + 8$

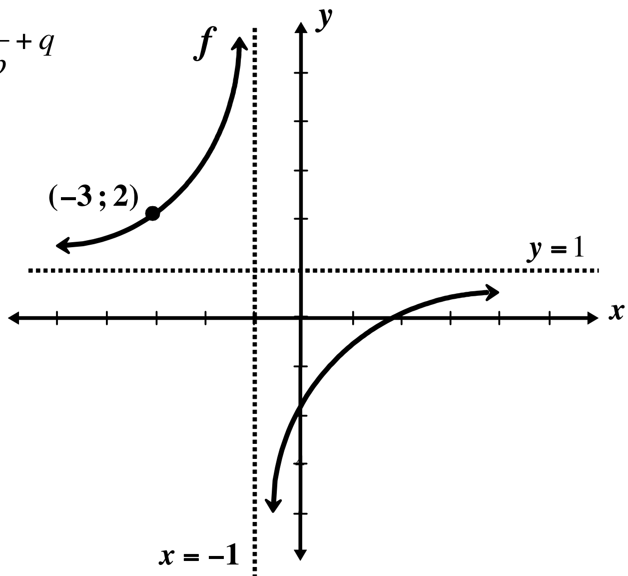
$\therefore y = -2x^2 - 4x - 2 + 8$

$\therefore y = -2x^2 - 4x + 6$

DETERMINING THE EQUATION OF A HYPERBOLA

EXAMPLE 17

Determine the equation of $f(x) = \dfrac{a}{x+p} + q$



Vertical asymptote: $x = -1$
$\therefore x + 1 = 0$
Therefore the expression $x + 1$ is in the denominator of the equation.

Horizontal asymptote: $y = 1$
Therefore the value of q is 1.

Therefore the equation becomes:

$$y = \frac{a}{x+1} + 1$$

All we need to now do is substitute the point $(-3\,;2)$ into the equation to get the value of a.

$\therefore 2 = \dfrac{a}{-3+1} + 1$

$\therefore 2 = \dfrac{a}{-2} + 1$

$\therefore 2 - 1 = \dfrac{a}{-2}$

$\therefore 1 = \dfrac{a}{-2}$

$\therefore a = -2$

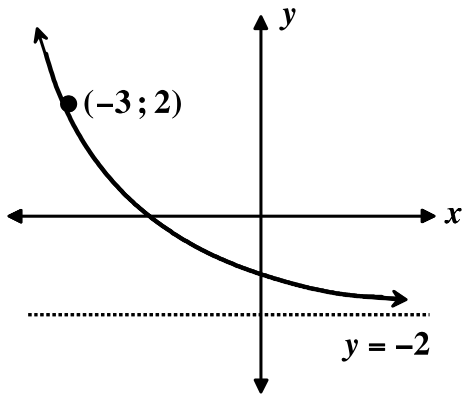
Therefore, substitute this value of a into the equation to get the final equation:

$y = \dfrac{-2}{x+1} + 1$

<u>DETERMINING THE EQUATION OF AN EXPONENTIAL GRAPH</u>

<u>EXAMPLE 18</u>

Determine the equation of $g(x) = b^{x+1} + q$



Horizontal asymptote: $y = -2$

Therefore the value of q is -2

$\therefore y = b^{x+1} - 2$

Now substitute the point $(-3\,;2)$ into the equation to get b.

$2 = b^{-3+1} - 2$

$\therefore 4 = b^{-2}$

$\therefore 4 = \dfrac{1}{b^2}$

$\therefore 4b^2 = 1$

$\therefore 4b^2 - 1 = 0$

$\therefore (2b+1)(2b-1) = 0$

$\therefore b = -\dfrac{1}{2}$ or $b = \dfrac{1}{2}$

But $b \neq -\dfrac{1}{2}$

$\therefore b = \dfrac{1}{2}$

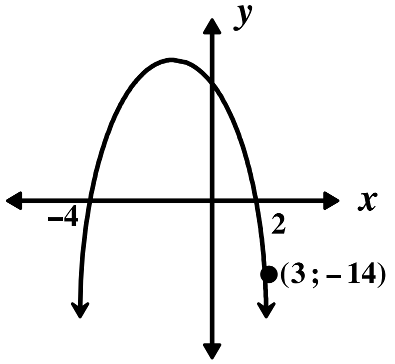
> Note:
> Since $b > 0$, you could have solved the equation $4b^2 = 1$ as follows:
>
> $4b^2 = 1$
>
> $\therefore b^2 = \dfrac{1}{4}$
>
> $\therefore b = \dfrac{1}{2}$

Therefore the equation is: $g(x) = \left(\dfrac{1}{2}\right)^{x+1} - 2$

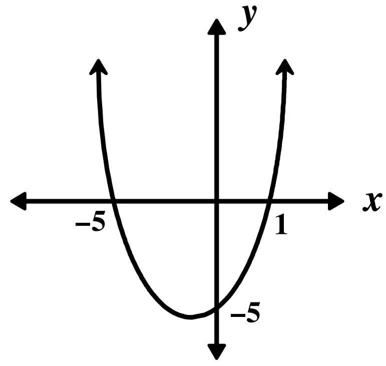
EXERCISE 7

1. Determine the equations of the following in the form $y = ax^2 + bx + c$:

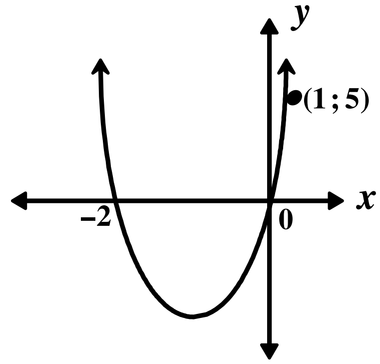
(a)



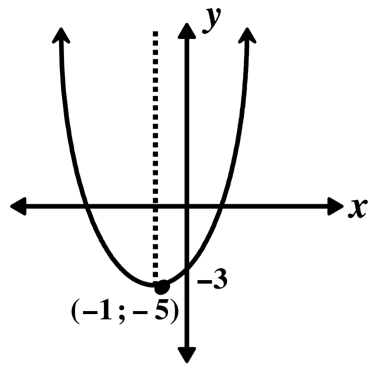
(b)



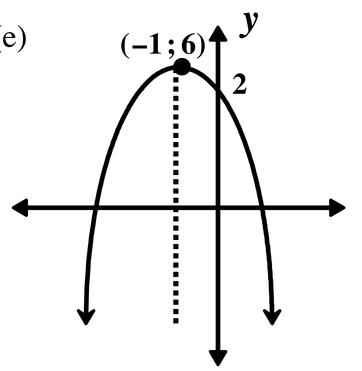
(c)



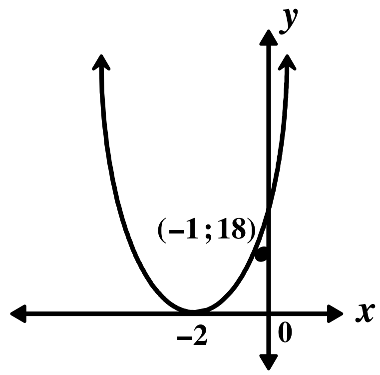
(d)



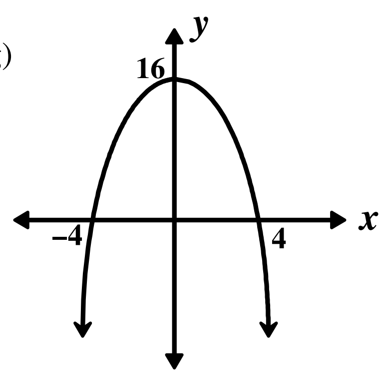
(e)



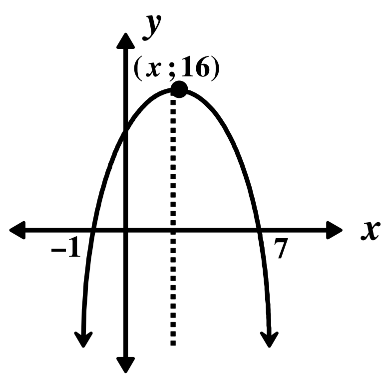
(f)



(g)

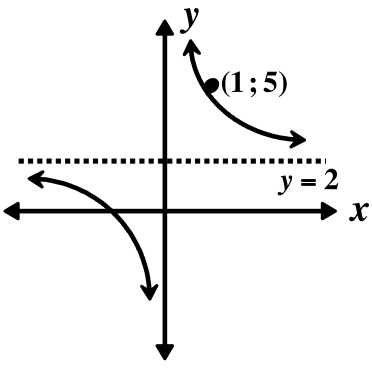


(h)

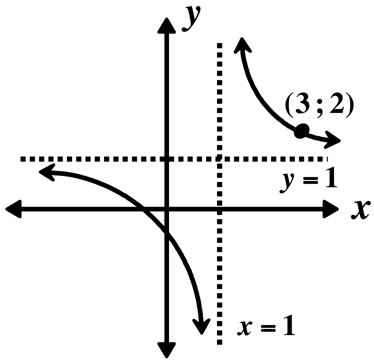


2. Determine the equations of the following graphs:

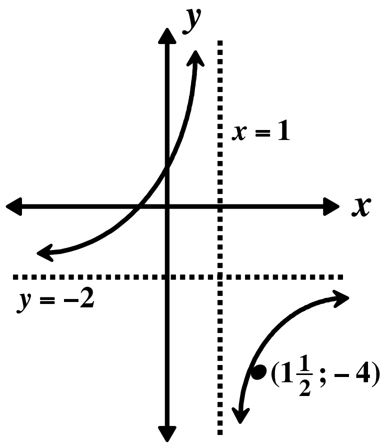
(a) $f(x) = \dfrac{a}{x} + q$



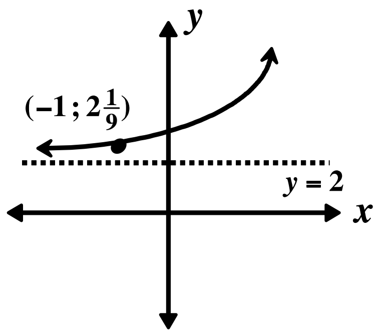
(b) $g(x) = \dfrac{a}{x+p} + q$



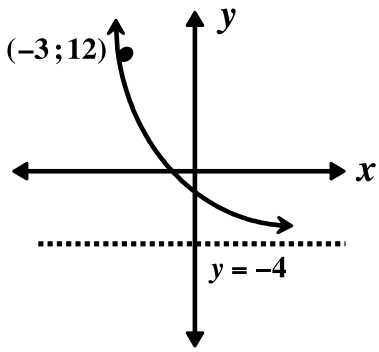
(c) $f(x) = \dfrac{a}{x+p} + q$



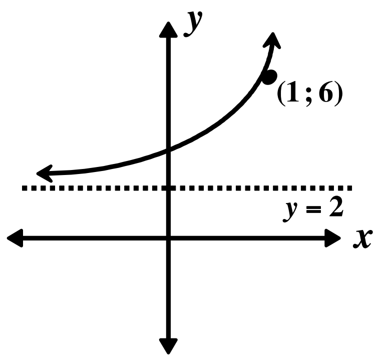
(d) $y = b^{x-1} + q$



(e) $y = b^{x+1} + q$



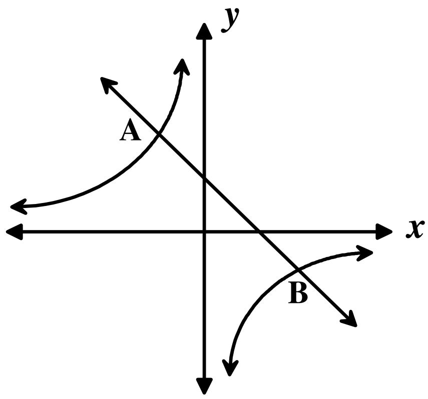
(f) $y = 2^{x+p} + q$



INTERSECTING GRAPHS

EXAMPLE 19

Determine the coordinates of A and B, the points of intersection of the graphs $xy = -6$ and $x + y - 1 = 0$.



$x + y - 1 = 0$

$\therefore y = -x + 1$

Substitute into the equation of the hyperbola:

$xy = -6$

$\therefore x(-x+1) = -6$

$\therefore -x^2 + x = -6$

$\therefore 0 = x^2 - x - 6$

$\therefore 0 = (x-3)(x+2)$

$\therefore x = 3 \quad \text{or} \quad x = -2$

For $x = 3$

$y = -3 + 1$

$\therefore y = -2$

For $x = -2$

$y = -(-2) + 1$

$\therefore y = 3$

The coordinates of the points of intersection are:

$\text{A}(-2\,;3)$ and $\text{B}(3\,;-2)$

EXERCISE 8

1. Determine the coordinates of the point(s) of intersection of the graphs:
   - (a) The parabola $f(x) = x^2 + 6x + 14$ and the line $g(x) = -3x - 4$.
   - (b) The parabola $f(x) = 2x^2 - 3x - 2$ and the line $x + y = 2$.
   - (c) The hyperbola $xy = 4$ and the line $x + y = 4$.
   - (d) The lines $2x - y = 7$ and $x - 2y = 8$.
   - (e) The exponential graph $y = 2^{x+1}$ and the line $y = 4$.

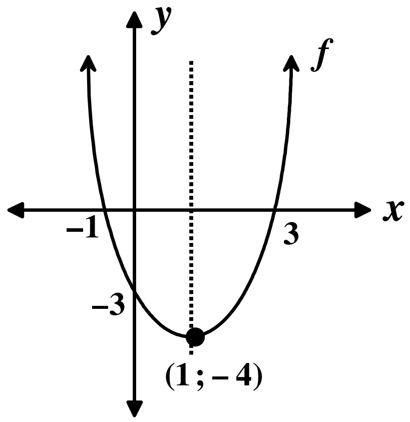
2. $f : x \rightarrow -x^2 + 10x + 24$ and $g : x \rightarrow 2x + 4$.
   - (a) Sketch the graphs of f and g on the same set of axes.
   - (b) Calculate the coordinates of the points of intersection of the two graphs and indicate these points on your diagram.
   - (c) Determine the domain and range of f and g.

FURTHER TRANSLATION TYPE EXAMPLES (PARABOLAS)

EXAMPLE 20

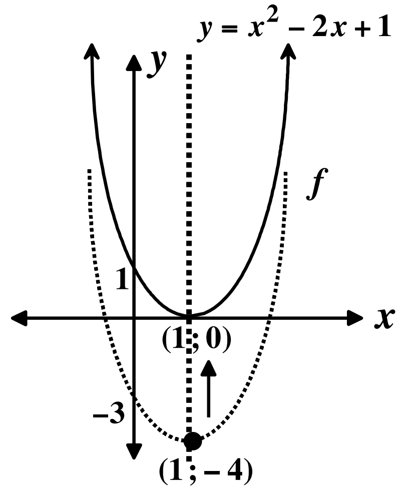
Consider the graph of $f(x) = x^2 - 2x - 3 = (x-1)^2 - 4$.
Determine graphically the value(s) of k for which:

(a) $x^2 - 2x + k = 0$ has equal solutions (roots)
(b) $x^2 - 2x + k = 0$ has non-real solutions (roots)
(c) $x^2 - 2x + k = 0$ has two real, unequal solutions
(d) $x^2 - 2x - 3 = k$ has equal solutions (roots)
(e) $x^2 - 2x - 3 = k$ has non-real solutions (roots)
(f) $x^2 - 2x - 3 = k$ has two real, unequal solutions

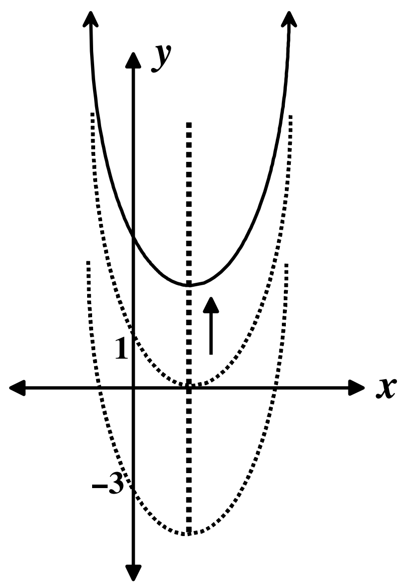


Solutions

(a) $x^2 - 2x + k = 0$ has equal solutions (roots)
If the graph of the given parabola is shifted 4 units upwards, the newly formed parabola will have equal x-intercepts.
The equation of the new parabola will be $y = x^2 - 2x - 3 + 4 = x^2 - 2x + 1$.
The y-intercept of this parabola is 1.
Since k represents the y-intercept of the parabola, it is clear that $k = 1$.

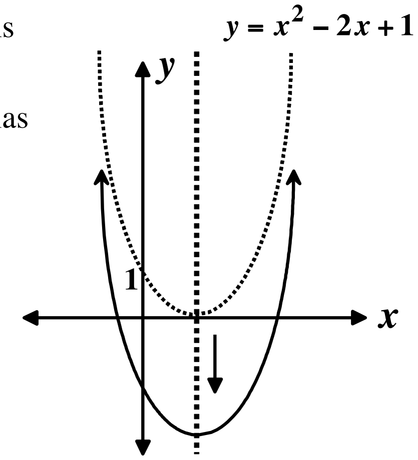


(b) $x^2 - 2x + k = 0$ has non-real solutions (roots)
If the graph of the parabola $y = x^2 - 2x + 1$ is shifted upwards, the newly formed parabolas will have no x-intercepts.
The y-intercepts of these parabolas will be greater than 1.
$\therefore k > 1$



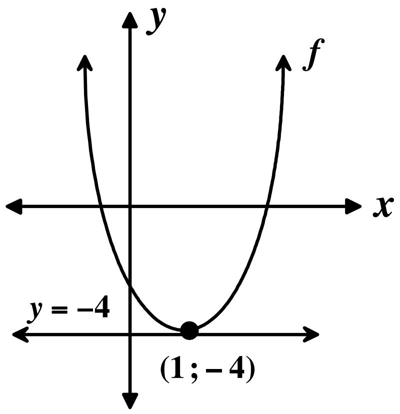
(c) $x^2 - 2x + k = 0$ has two real, unequal solutions

If the graph of the parabola $y = x^2 - 2x + 1$ is shifted downwards, the newly formed parabolas will have two unequal x-intercepts.
The y-intercepts of these parabolas will be less than 1.
$\therefore k < 1$



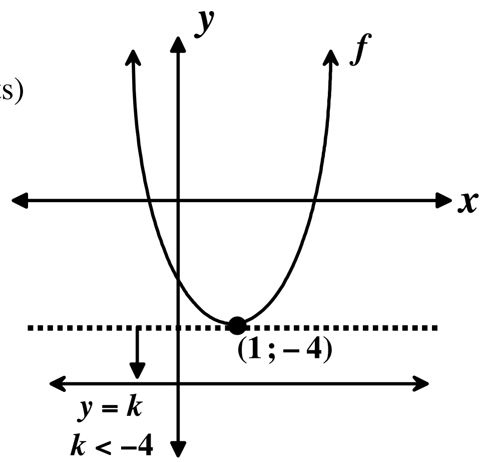
(d) $x^2 - 2x - 3 = k$ has equal solutions (roots)

The equation $x^2 - 2x - 3 = k$ involves the intersection of the graph $y = x^2 - 2x - 3$ with the moving horizontal line $y = k$.
The moving horizontal line $y = k$ will intersect the parabola in two unequal points, touch the parabola at one point only (equal solutions) or not intersect with the parabola at all.
Therefore the intersection equation $x^2 - 2x - 3 = k$ will have equal solutions at the turning point of the parabola.
The horizontal line at the turning point is the line $y = -4$
$\therefore k = -4$



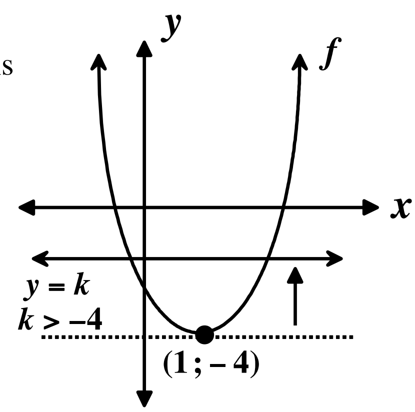
(e) $x^2 - 2x - 3 = k$ has non-real solutions (roots)

The intersection equation $x^2 - 2x - 3 = k$ will have no real solutions for $k < -4$.
The horizontal line $y = k$ doesn't intersect with the parabola for $k < -4$



(f) $x^2 - 2x - 3 = k$ has two real, unequal solutions

The intersection equation $x^2 - 2x - 3 = k$ will have two real solutions for $k > -4$.
The horizontal line $y = k$ will intersect with the parabola in two distinct points.



EXERCISE 9

1. Sketch the graph of $f(x) = -x^2 - 2x + 8$.
   Determine graphically the value(s) of k for which:
   (a) $-x^2 - 2x + k = 0$ has real and equal solutions
   (b) $-x^2 - 2x + k = 0$ has non-real solutions
   (c) $-x^2 - 2x + k = 0$ has real and unequal solutions
   (d) $-x^2 - 2x + k = 0$ has two unequal, negative solutions
   (e) $-x^2 - 2x + 8 = k$ has non-real solutions
   (f) $-x^2 - 2x + 8 = k$ has equal solutions
   (g) $-x^2 - 2x + 8 = k$ has two distinct real solutions
   (h) $-x^2 - 2x + 8 = k$ has two unequal solutions which differ in sign

2. Consider $f(x) = 2(x+1)^2 - 2$
   (a) Sketch the graph of f.
   (b) Use the graph to determine graphically the values of k such that:
      (1) $2x^2 + 4x + k = 0$ will have two unequal, negative solutions.
      (2) $2(x+1)^2 + k = 0$ will have non-real solutions.

AVERAGE GRADIENT BETWEEN TWO POINTS

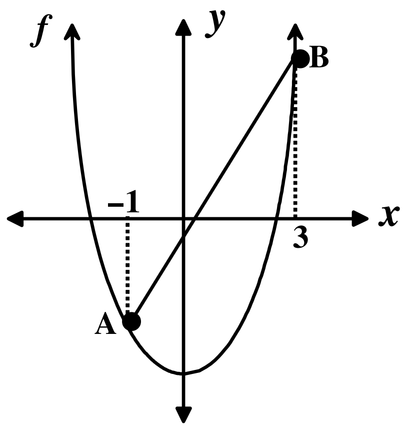
The **average gradient** of a function between any two points is defined to be the **gradient of the line** joining the two points.
Note to the educator:
The assignment in the Teacher's Guide explores the concept of average gradient in more detail.

EXAMPLE 21

Determine the average gradient of the graph of $f(x) = x^2 - 4$ between $x = -1$ and $x = 3$.

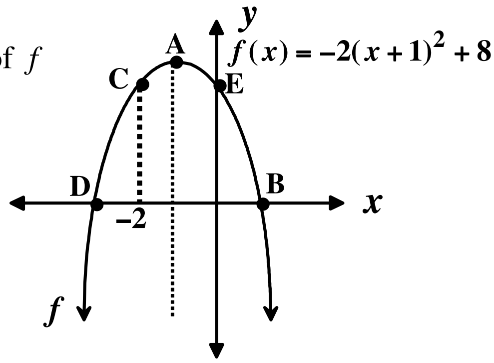
The average gradient of f between $x = -1$ and $x = 3$ is the gradient of the line AB which joins these two points on the graph. We need to determine the y-coordinates corresponding to the given x-values. Then it will be easy to determine the gradient of line segment AB. This gradient will be the average gradient of the parabola between the given x-values.

For $x = -1$
$y = (-1)^2 - 4 = -3$
$A(-1\,;-3)$
For $x = 3$
$y = (3)^2 - 4 = 5$
$B(3\,;5)$
Therefore the gradient of AB is:
$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{5-(-3)}{3-(-1)} = \frac{8}{4} = 2$$
Therefore the **average gradient** of f between the given x-values is 2.

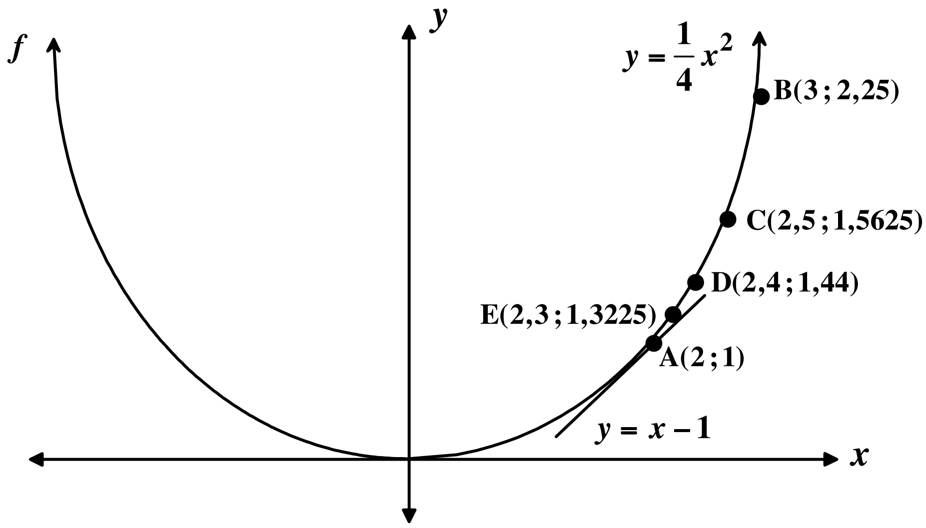


EXERCISE 10

1. Determine the average gradient of f between:
   (a) A and B
   (b) B and C
   (c) D and E



2. In the diagram below, the graph of $f(x) = \frac{1}{4}x^2$ as well as the tangent line to f at $x = 2$ is shown. The equation of the tangent line is $y = x - 1$.
   Some points in the interval $[2\,;3]$ are shown on the graph of f.



   (a) Calculate the average gradient of f between:
      (1) A and B
      (2) A and C
      (3) A and D
      (4) A and E
   (b) To which value does the average gradient tend to as the points on the graph of f get closer and closer to A?
   (c) What is the gradient of the tangent line to f at $x = 2$?
   (d) What can you conclude?

Note: The gradient of the tangent line to the graph of f at $x = 2$ is defined as the gradient of the graph of f at $x = 2$.
This concept will be explored further in Grade 12.

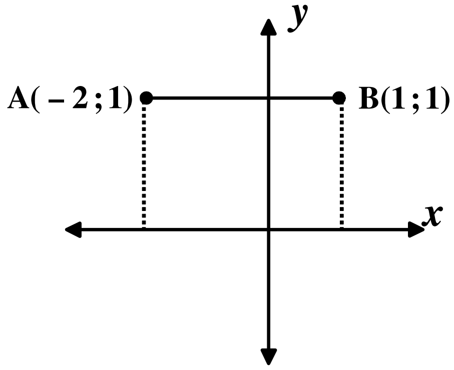
DEDUCTIONS BASED ON GRAPHS

HORIZONTAL AND VERTICAL LENGTHS

The **horizontal length** of AB can be determined by using the fact that

$AB = x_B - x_A$ (largest x minus smallest x)
In this example:

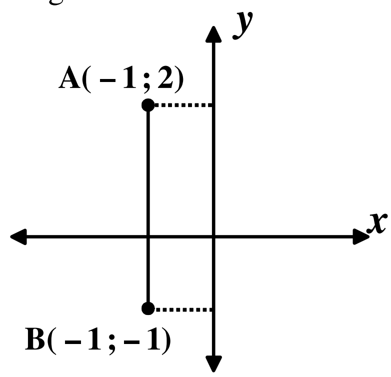
$AB = 1 - (-2)$
$\therefore AB = 3$



The **vertical length** of AB can be determined by using the fact that

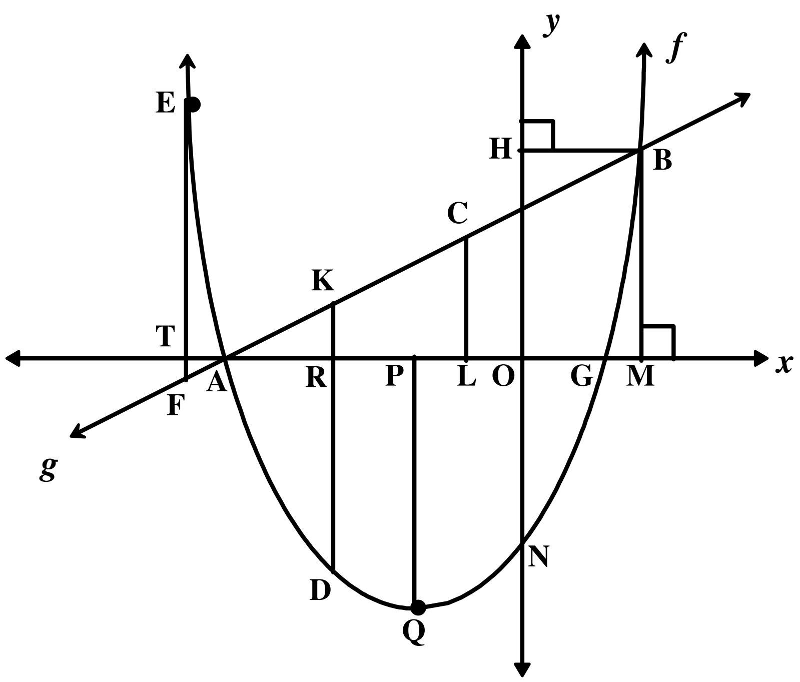
$AB = y_A - y_B$ (largest y minus smallest y)

In this example:
$AB = 2 - (-1)$
$\therefore AB = 3$



EXAMPLE 22

In the diagram below are the graphs of $f(x) = x^2 + 3x - 4$ and $g(x) = x + 4$.



(a) Determine the length of AG and ON.

A and G are the x-intercepts of the parabola.

$\therefore 0 = x^2 + 3x - 4$

$\therefore 0 = (x+4)(x-1)$

$\therefore x = -4 \quad \text{or} \quad x = 1$

$\therefore$ AG = 5 units

N is the y-intercept of the parabola.

$\therefore$ ON = 4 units

(b) Determine the length of CL if $\text{OL} = 1$ unit

Since $\text{OL} = 1$ unit, it is clear that $x_L = -1$

Substitute $x = -1$ into the equation of the line to get the y-coordinate of C.

$\therefore y = -1 + 4 = 3$

$\therefore$ CL = 3 units

(c) Determine the length of OT if $\text{ET} = 6$ units

Since $\text{ET} = 6$ units, it is clear that $y_E = 6$

Substitute $y = 6$ into the equation of the parabola to get the x-coordinate of T.

$6 = x^2 + 3x - 4$

$\therefore 0 = x^2 + 3x - 10$

$\therefore 0 = (x+5)(x-2)$

$\therefore x = -5 \text{ or } x = 2$

Clearly, from the diagram, $x_T = -5$. Therefore $\text{OT} = 5$ units.

(d) Determine the length of KD if $\text{OR} = 2$ units

Since $\text{OR} = 2$ units, it is clear that $x_R = -2$

We now need to get the length of KD.

Since KD is a vertical length, $\text{KD} = y_K - y_D$.

$\therefore \text{KD} = (x+4) - (x^2 + 3x - 4)$

$\therefore \text{KD} = x + 4 - x^2 - 3x + 4$

$\therefore \text{KD} = -x^2 - 2x + 8$

The length of KD at $x = -2$ is

$\text{KD} = -(-2)^2 - 2(-2) + 8 = -4 + 4 + 8 = 8$ units.

(e) Determine the length of OT if $\text{EF} = 7$ units

Since EF is a vertical length, $\text{EF} = y_E - y_F$

$\therefore \text{EF} = (x^2 + 3x - 4) - (x+4)$

$\therefore 7 = x^2 + 3x - 4 - x - 4 \qquad (\text{since EF} = 7)$

$\therefore 0 = x^2 + 2x - 15$

$\therefore 0 = (x+5)(x-3)$

$\therefore x = -5 \text{ or } x = 3$

From the diagram, it is clear that $x_T = -5$

$\therefore \text{OT} = 5$ units.

(f) Determine the length of BM and HB.

To get these lengths, it is first necessary to determine the coordinates of B, the point of intersection of the line and the parabola.

$\therefore x^2 + 3x - 4 = x + 4$

$\therefore x^2 + 2x - 8 = 0$

$\therefore (x+4)(x-2) = 0$

$\therefore x = -4$ or $x = 2$

Clearly from the diagram, $x_B = 2$

To get the y-coordinate of B, substitute into either the equation of the line or parabola.

$\therefore y = 2 + 4 = 6$

Therefore, B is the point $(2\,;6)$.

$\therefore$ BM = 6 units and HB = 2 units

(g) Determine the coordinates of Q, the turning point.

Here we can use the formula $x = -\dfrac{b}{2a}$ to get the x-intercept of the turning point of the parabola $y = x^2 + 3x - 4$

Since $a = 1$ and $b = 3$, $x_Q = -\dfrac{3}{2(1)} = -\dfrac{3}{2} = -1\dfrac{1}{2}$

Substitute this value into the equation to get the y-coordinate of the turning point.

$\therefore y = \left(-\dfrac{3}{2}\right)^2 + 3\left(-\dfrac{3}{2}\right) - 4$

$\therefore y = \dfrac{9}{4} - \dfrac{9}{2} - 4 = -\dfrac{25}{4} = -6\dfrac{1}{4}$

$\therefore$ The turning point is $Q\left(-\dfrac{3}{2}\,;-6\dfrac{1}{4}\right)$

(h) Determine the length of PQ.

$PQ = 6\dfrac{1}{4}$ units

(i) Determine the maximum length of KD.

To find the maximum length of the line segment KD, we use completing the square.

$KD = (x+4) - (x^2 + 3x - 4)$

$\therefore KD = x + 4 - x^2 - 3x + 4$

$\therefore KD = -x^2 - 2x + 8$

$\therefore KD = -(x^2 + 2x) + 8$

$\therefore \text{KD} = -\left[x^2 + 2x + \left(\frac{1}{2}(2)\right)^2 - \left(\frac{1}{2}(2)\right)^2 \right] + 8$

$\therefore \text{KD} = -\left[x^2 + 2x + 1 - 1 \right] + 8$

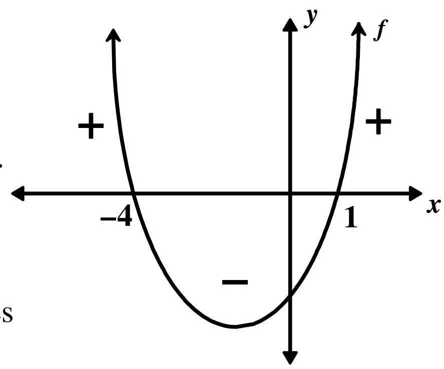
$\therefore \text{KD} = -\left[(x+1)^2 - 1 \right] + 8$

$\therefore \text{KD} = -(x+1)^2 + 9$

The expression $-(x+1)^2 + 9$ represents a parabola in its own right. It has a maximum value at its turning point $(-1\,;9)$ since its shape is "unhappy". The y-coordinate of the turning point is the maximum value of the parabola.
Therefore, the maximum length of KD is 9 units.

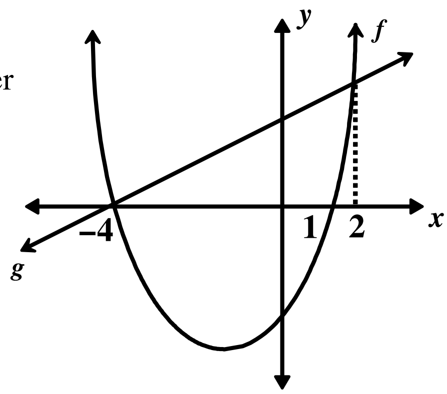
(j) Determine graphically the values of x for which:

(1) $f(x) > 0$
Here we need to determine the values of x for which the y-values of the parabola are positive (where parabola is above the x-axis).
The solution is: $x < -4$ or $x > 1$



(2) $f(x) \leq 0$
Here we need to determine the values of x for which the y-values of the parabola are zero or negative (where parabola cuts or is below the x-axis).
The solution is: $-4 \leq x \leq 1$

(3) $f(x) = g(x)$
The parabola and line intersect each other at $x = -4$ or $x = 2$

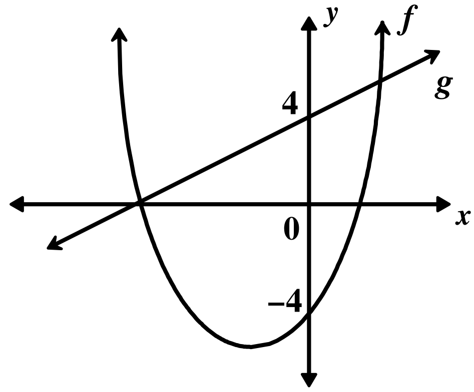


(4) $f(x) \geq g(x)$
Here we need to determine the values of x for which the y-values of the parabola are greater than or equal to the y-values of the line (where the parabola is above the line or intersects the line).
The solution is: $x \leq -4$ or $x \geq 2$

(5) $f(x) < g(x)$
Here we need to determine the values of x for which the y-values of the parabola are less than the y-values of the line (where the parabola is below the line).
The solution is: $-4 < x < 2$

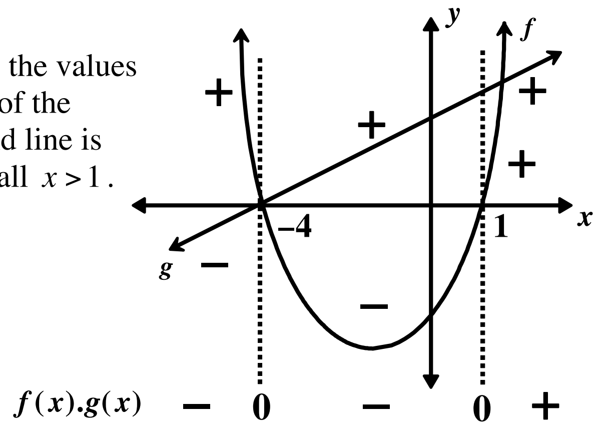
(6) $g(x) - f(x) = 8$

Here we need to determine the value of x for which a y-value on the line minus a y-value on the parabola equals 8.
This happens at $x = 0$ since $g(0) - f(0) = 4 - (-4) = 8$.
In other words, the difference between the y-intercepts is 8.



(7) $f(x).g(x) > 0$

Here we need to determine the values of x for which the product of the y-values of the parabola and line is positive. This happens for all $x > 1$.



(8) $f(x).g(x) \geq 0$

Here we need to determine the values of x for which the product of the y-values of the parabola and line is positive or zero.
This happens for $x = -4$ or $x \geq 1$.

(9) $f(x).g(x) < 0$

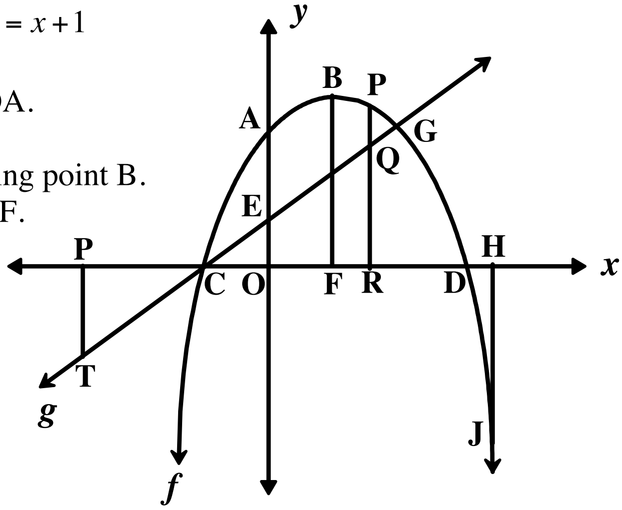
Here we need to determine the values of x for which the product of the y-values of the parabola and line is negative.
This happens for $x < -4$ or $-4 < x < 1$.

(10) $f(x).g(x) \leq 0$

Here we need to determine the values of x for which the product of the y-values of the parabola and line is negative or zero.
This happens for $x \leq 1$.

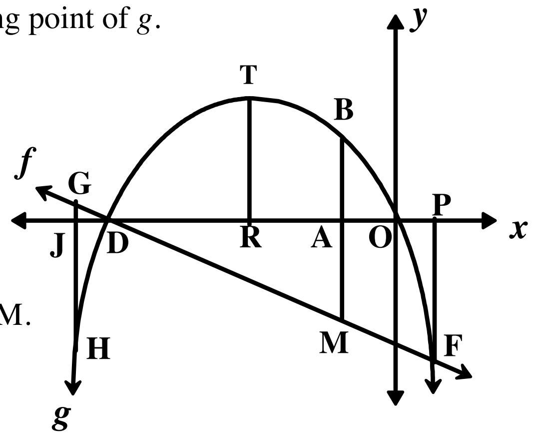
EXERCISE 11

1. The sketch shows the graphs of two functions
$f(x) = -x^2 + 2x + 3$ and $g(x) = x + 1$



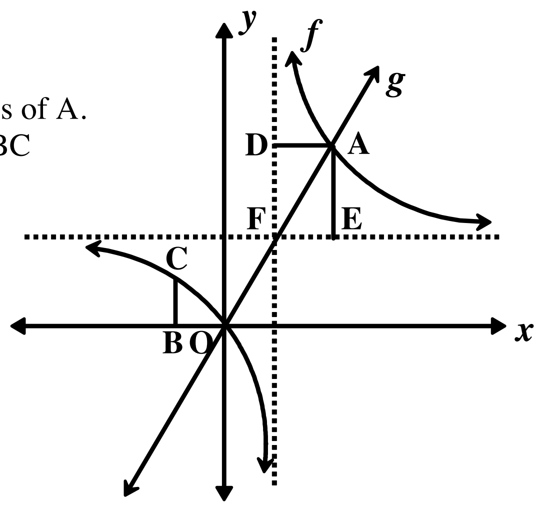
Determine:
   (a) the length of OE and OA.
   (b) the length of CD.
   (c) the coordinates of turning point B.
   (d) the length of OF and BF.
   (e) the length of PR if $OR = 1\frac{1}{2}$ units .
   (f) the length of PT if $OP = 5$ units .
   (g) the length of OH if $HJ = 5$ units .
   (h) the length of PQ if $OR = 1\frac{1}{2}$ units .
   (i) the maximum length of PQ.
   (j) the coordinates of G.

2. The graphs of $f : x \to -2x - 8$ and $g : x \to -2x^2 - 8x$ are represented in the diagram below. T is the turning point of g.



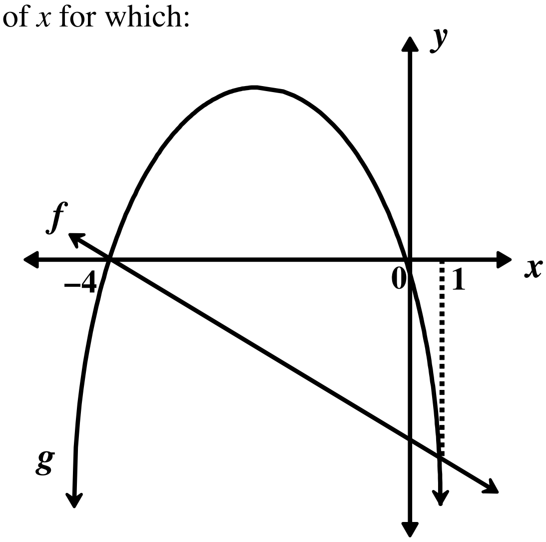
Determine:
   (a) the length of OD.
   (b) the length of TR.
   (c) the equation of TR.
   (d) BM if $OA = 1$.
   (e) OJ if $GH = 28$.
   (f) the length of FP.
   (g) the maximum length of BM.

3. The graphs of $f(x) = \dfrac{2}{x-1} + 2$ and $g(x) = 2x$ are shown.



   (a) Determine the equations of the asymptotes of f.
   (b) Determine the coordinates of A.
   (c) Determine the length of BC if $OB = 1$ unit .
   (d) Determine the area of rectangle ADFE.

4. The graphs of $f : x \rightarrow -2x - 8$ and $g : x \rightarrow -2x^2 - 8x$ are represented in the diagram below.
Determine graphically the values of x for which:



(a) $f(x) > 0$
(b) $f(x) \leq 0$
(c) $f(x) = g(x)$
(d) $f(x) \geq g(x)$
(e) $f(x) < g(x)$
(f) $g(x) - f(x) = 8$
(g) $g(x) - f(x) = 12$
(h) $f(x).g(x) \geq 0$
(i) $f(x).g(x) > 0$
(j) $f(x).g(x) \leq 0$
(k) $f(x).g(x) < 0$

REVISION EXERCISE

1. (a) Consider the function $f(x) = (x+1)^2 - 4$
   (1) Draw a neat sketch graph indicating the coordinates of the intercepts with the axes, the coordinates of the turning point and the equation of the axis of symmetry.
   (2) Write down the domain and range.
   (3) Determine the values of x for which the graph increases.

   (b) Consider the function $g(x) = -x^2 + 4x - 3$
   (1) Determine the maximum value of g.
   (2) Determine the coordinates of the turning point of the graph of $g(x+1)$

2. Given: $f(x) = \dfrac{2}{x+1}$
   (a) Write down the equations of the asymptotes.
   (b) Sketch the graph of f indicating the coordinates of the y-intercept as well as the asymptotes.
   (c) Write down the equation of the graph formed if the graph of f is shifted 3 units right and 2 units upwards.
   (d) Determine graphically the values of x for which
   (1) $f(x) > 0$ (2) $\dfrac{2}{x+1} \geq 1$

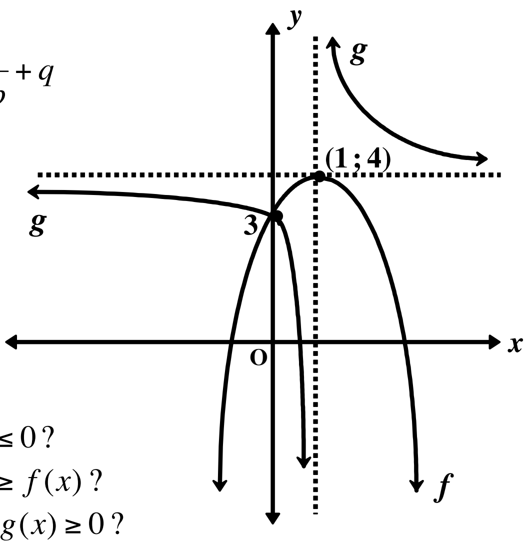
3. Given: $f(x) = \left(\dfrac{1}{2}\right)^{x+1}$ and $g(x) = (x+2)^2 + 2$
   (a) Sketch the graphs of f and g on the same set of axes.
   (b) Write down the domain and range of f and g.
   (c) Write down the equation of the horizontal asymptote of f.
   (d) Write down the equation of the axis of symmetry of g.
   (e) Determine the minimum value of the graph of $h(x) = g(x-2)$.
   (f) Determine the values of x for which the graph of g decreases.

4. In the diagram below, the graphs of the following functions are represented:

$f(x) = a(x+p)^2 + q$ and $g(x) = \dfrac{a}{x+p} + q$

The turning point of f is $(1\,;4)$ and the asymptotes of g intersect at the turning point of f.

Both graphs cut the y-axis at 3.



(a) Determine the equation of f.

(b) Determine the equation of g.

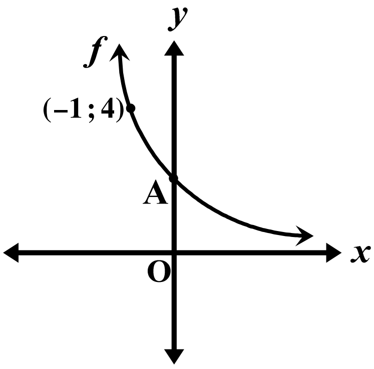
(c) Determine the coordinates of the x-intercept of g.

(d) For which values of x will $g(x) \le 0$?

(e) For which values of x will $g(x) \ge f(x)$?

(f) For which values of x will $f(x).g(x) \ge 0$?

5. In the diagram below (not drawn to scale), the graph of $y = f(x) = 2a^x$ is sketched. The graph of f passes through the point $(-1\,;4)$ and cuts the y-axis at A.



(a) Show that $a = \frac{1}{2}$ and hence write down the equation of f.

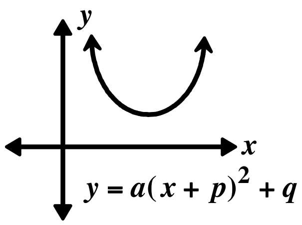
(b) Determine the coordinates of A.

(c) Show that the equation of g, the reflection of f about the y-axis, can be written as $g(x) = 2^{x+1}$.

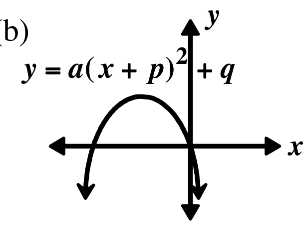
(d) Draw a neat sketch graph of $y = f(x-1) - 2$ indicating the intercepts with the axes as well as the equation of the asymptote.

6. For each of the functions below, state whether the sign of a, p and q is positive, negative or zero.

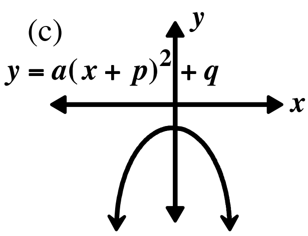
(a)



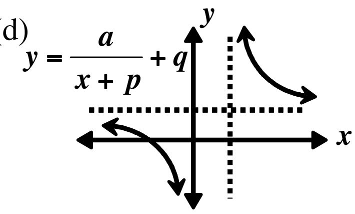
(b)



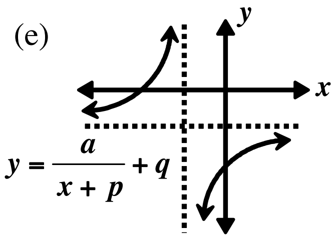
(c)



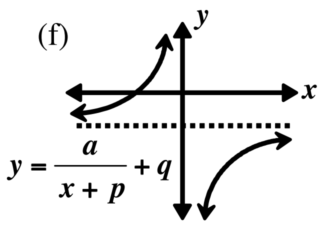
(d)



(e)



(f)

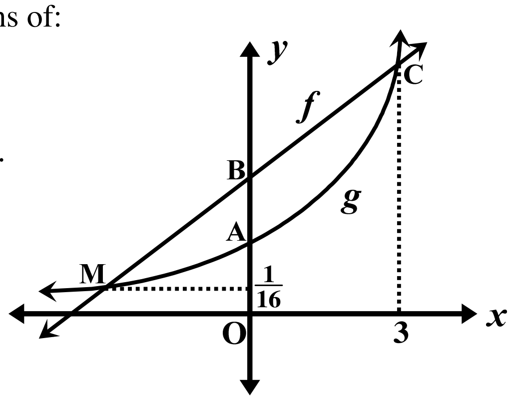


SOME CHALLENGES

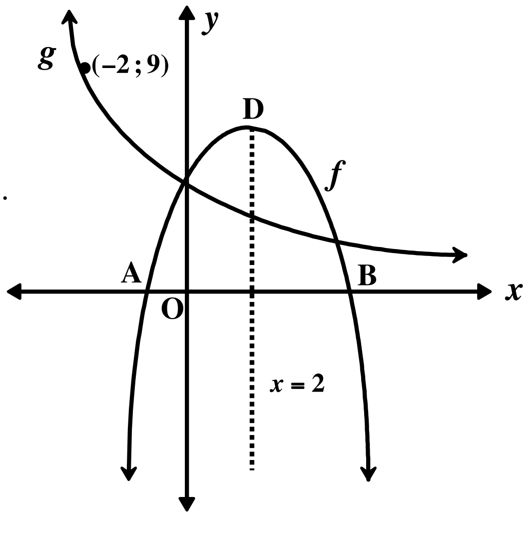
1. The line $y = ax - 3$ and the hyperbola $y = \dfrac{5}{2x+3}$ intersect at $x = 1$ and $x = b$.
   - (a) Determine the y-coordinate of the point of intersection if $x = 1$.
   - (b) Determine the value of a.
   - (c) Determine the value of b.

2. Given: $f(x) = \dfrac{x+1}{x-1}$
   - (a) Show that the graph of f is a hyperbola with a horizontal asymptote of $y = 1$.
   - (b) Determine the equation of the vertical asymptote.
   - (c) Sketch the graph of $y = f(x)$.

3. The diagram below represents the graphs of:
   $f(x) = \frac{127}{112}x + k$ and $g(x) = 2^x$
   - (a) Determine the coordinates of C.
   - (b) Determine the coordinates of M.
   - (c) Determine the length of AB. (one decimal place)
   - (d) Determine the average gradient of g between M and C.



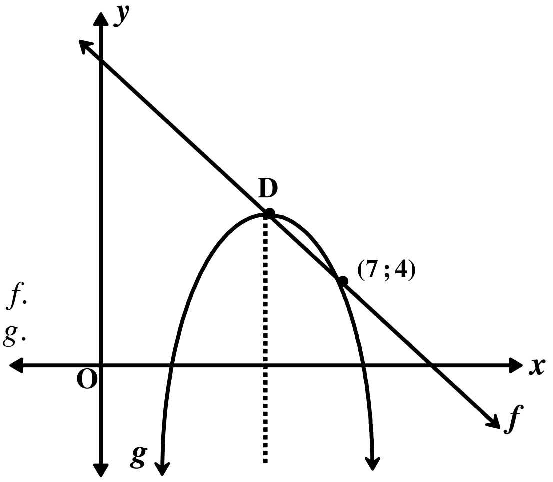
4. The sketch represents the graphs of $f(x) = ax^2 + bx + c$ and $g(x) = k^x$
   The equation of the axis of symmetry of f is $x = 2$ and the point $(-2\,;9)$ lies on g. The length of AB is 6 units. A and B are the x-intercepts of f and D is the turning point.
   - (a) Determine the value of k.
   - (b) Determine the coordinates of A and B.
   - (c) Prove that $a = -\frac{1}{5}$ and $b = \frac{4}{5}$.
   - (d) Determine the coordinates of D.
   - (e) Use the graph to determine the values of p for which $-\frac{1}{5}x^2 + \frac{4}{5}x + p < 0$.



5. The straight line $f(x) = ax + 32$ passes through the turning point of the parabola with equation $g(x) = a(x + p)^2 + 12$. The line and parabola intersect at the point $(7\,;4)$.

(a) Determine the equation of f.

(b) Determine the equation of g.



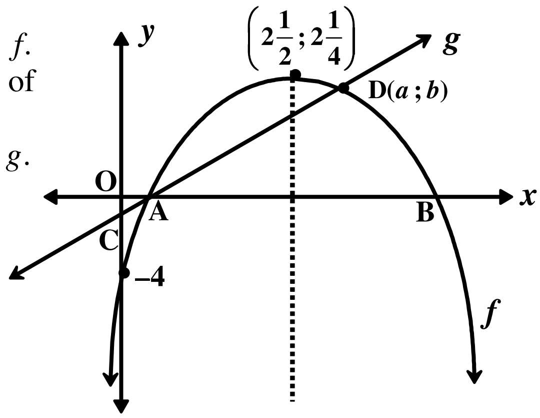
6. In the diagram, f is a parabola with turning point and y-intercept indicated. The parabola intersects the x-axis at A and B. C is a point on the y-axis, symmetrical to A with respect to the line $y = -x$ and $D(a\,;b)$ is a point of intersection between the straight line g and the parabola. C is the y-intercept of g.

(a) Determine the equation of f.

(b) Determine the coordinates of A, B and C.

(c) Determine the equation of g.

(d) Determine the value of a and b.



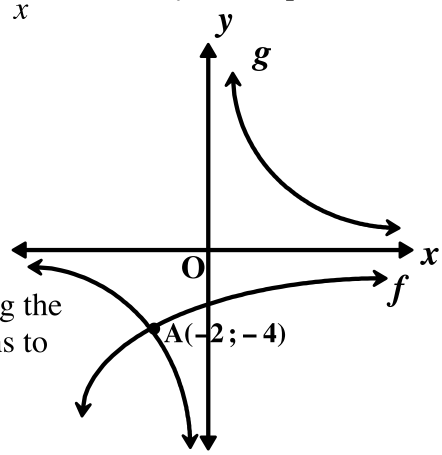
7. In the diagram, the graph of f is symmetrical about the x-axis to the curve $y = \left(\frac{1}{2}\right)^x$. The hyperbola g with equation $y = \frac{k}{x}$ intersects f at the point $A(-2\,;-4)$.

(a) Determine the equation of f.

(b) Determine the equation of g.

(c) The equation $\frac{16}{x} + 2^{1-x} = 0$ has one real solution.

Express this equation in a form involving the equations of f and g and use the graphs to find the solution of the equation.



8. Draw a sketch graph of $y = ax^2 + bx + c$ if

(a) $a > 0 \quad b < 0 \quad c > 0$

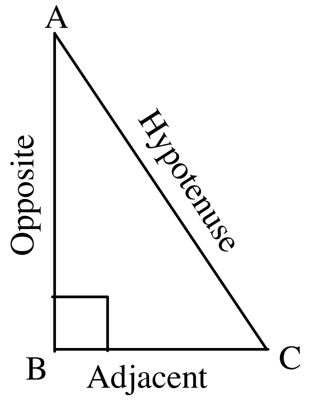
(b) $a < 0 \quad b < 0 \quad c < 0$

CHAPTER 6 – TRIGONOMETRY

In Grade 10, the trigonometric functions **sine of** θ **, cosine of** θ and the **tangent of** θ **,** written respectively as $\sin\theta$, $\cos\theta$ and $\tan\theta$ where θ is an angle, were defined.

In any **right-angled triangle:**

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{and} \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



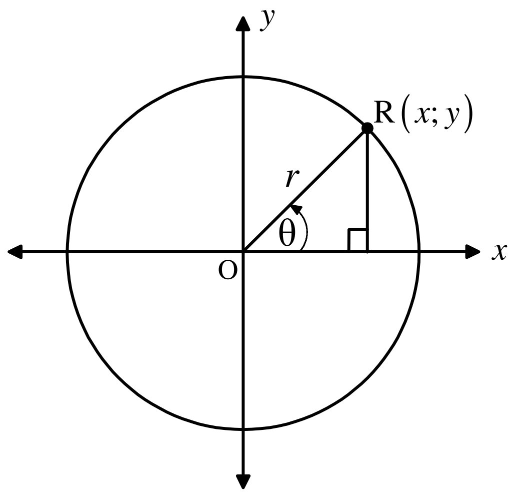
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{AB}{AC}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{BC}{AC}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{AB}{BC}$$

When **angles on a Cartesian plane** are formed then:

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

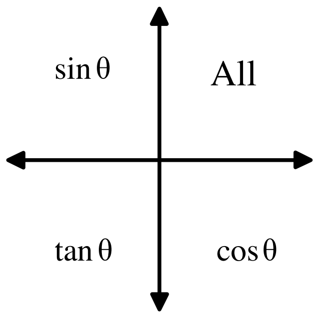


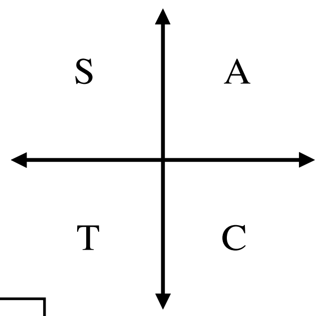
.

θ is the angle measured in an anti-clockwise direction from the positive side of the x –axis. Also note that $x^2 + y^2 = r^2$

The Cartesian plane is divided into four quadrants. The angle formed will determine the sign of each trigonometric function.

- All trigonometric functions are positive in the first quadrant.
- $\sin\theta$ is positive in the second quadrant and $\tan\theta$ and $\cos\theta$ are negative.
- $\tan\theta$ is positive in the third quadrant and $\sin\theta$ and $\cos\theta$ are negative.
- $\cos\theta$ is positive in the fourth quadrant and $\sin\theta$ and $\tan\theta$ are negative.





This is also referred to as the CAST diagram

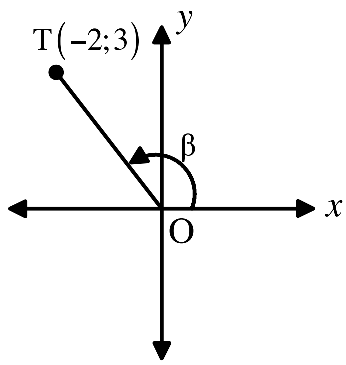
REVISION EXAMPLES

(a) Consider the diagram alongside. Point $T(-2;3)$ is a point on the Cartesian Plane such that β is the angle of inclination of OT. Calculate the following without the use of a calculator.

(1) $\tan\beta$

(2) $\sin^2\beta + \cos^2\beta$

(3) $13\sin\beta.\cos\beta$



Solution

$x^2 + y^2 = r^2$ ….Pythag.

$x = -2$ and $y = 3$

$(-2)^2 + (3)^2 = r^2$

$\therefore 4 + 9 = r^2$

$\therefore 13 = r^2$

$\therefore \pm\sqrt{13} = r$

But r is always positive

$\therefore r = \sqrt{13}$

$x = -2$
$y = 3$
$r = \sqrt{13}$

(1) $\tan\beta = \frac{3}{-2} = -\frac{3}{2}$

(2) $\sin^2\beta + \cos^2\beta$

$= \left(\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{-2}{\sqrt{13}}\right)^2$

$= \frac{9}{13} + \frac{4}{13}$

$= \frac{13}{13} = 1$

(3) $13\sin\beta.\cos\beta$

$= 13\left(\frac{3}{\sqrt{13}}\right)\left(\frac{-2}{\sqrt{13}}\right)$

$= 13\left(\frac{-6}{13}\right) = -6$

(b) If $17\cos\theta = -8$ and $\theta \in (180°;360°)$ calculate without the use of a calculator and with the aid of a diagram the value of $8\tan\theta - 17\sin\theta$.

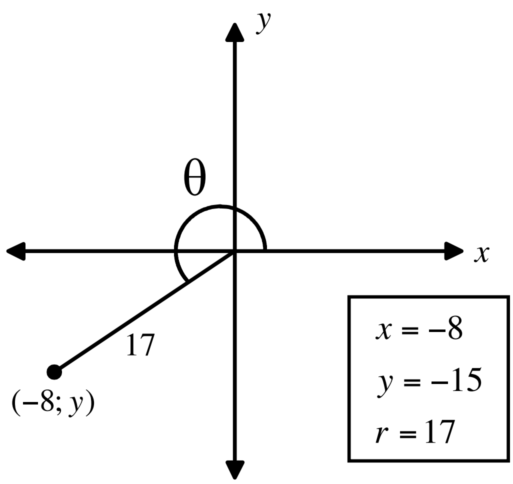
Solution

Given: $17\cos\theta = -8$

$\therefore \cos\theta = -\frac{8}{17} = \frac{x}{r}$ and r is always positive

$\cos\theta$ is negative and therefore the terminal arm will lie in the **second or third** quadrant.

But with $\theta \in (180°;360°)$, the terminal arm will lie in the **third** quadrant.



$x^2 + y^2 = r^2$ ....Pythagoras.

$\therefore y^2 = r^2 - x^2$

$x = -8$ and $r = 17$

$\therefore y^2 = (17)^2 - (-8)^2$

$\therefore y^2 = 289 - 64$

$\therefore y^2 = 225$

$\therefore y = \pm 15$

But y is negative in the third quadrant

$\therefore y = -15$

$\therefore 8\tan\theta - 17\sin\theta$

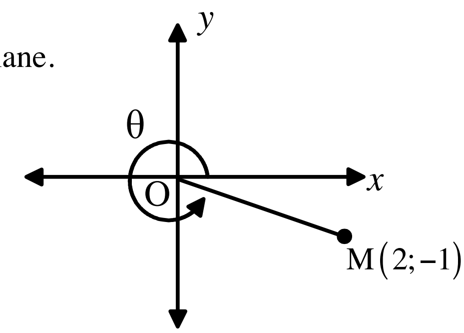
$= 8\left(\frac{-15}{-8}\right) - 17\left(\frac{-15}{17}\right)$

$= 15 + 15$

$= 30$

REVISION EXERCISE

1. Point $M(2;-1)$ is a point on the Cartesian Plane. Such that $X\hat{O}M = \theta$, a reflex angle. Calculate the following without the use of a calculator.
   (a) $\cos\theta$
   (b) $1 - \sin^2\theta$



2. If $13\cos\theta = 12$ and $180° < \theta < 360°$ calculate without the use of a calculator and with the aid of a diagram the value of
   (a) $\tan\theta$ (b) $(\sin\theta + \cos\theta)^2$

3. If $3\tan\theta - 4 = 0$ and $\cos\theta < 0$ calculate without the use of a calculator and with the aid of a diagram the value of

(a) $\dfrac{\sin\theta}{\cos\theta}$ (b) $10\sin\theta - 25\cos^2\theta$

4. If $\sin A = \dfrac{2\sqrt{6}}{5}$ and $A \in [90°;360°]$ calculate without the use of a calculator and with the aid of a diagram the value of $15\tan A.\cos A$

*5. If $\sin\alpha = \dfrac{3}{5}$ with $\alpha \in [90°;270°]$ and $\cos\beta = \dfrac{-12}{13}$ with $\beta \in [0°;180°]$ calculate without the use of a calculator and with the aid of a diagram the value of $\cos\alpha + \tan\beta$.

REDUCTION FORMULAE

Reduction formulae are used to **reduce** the trigonometric ratio of any angle to the trigonometric ratio of an acute angle. The formulae you will use are $180° \pm \theta$, $360° \pm \theta$, $90° \pm \theta$ and $-\theta$

Function values of $180° \pm \theta$ and $360° - \theta$

Consider the following table:

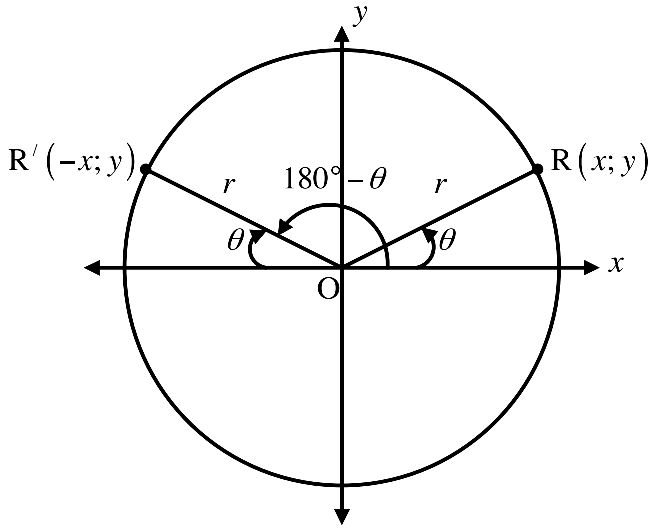
$\sin 30°$ $= 0,5$	$\sin(180° - 30°)$ $= \sin 150°$ $= 0,5$ $\therefore \sin(180° - 30°) = \sin 30°$	$\sin(180° + 30°)$ $= \sin 210°$ $= -0,5$ $\therefore \sin(180° + 30°) = -\sin 30°$	$\sin(360° - 30°)$ $= \sin 330°$ $= -0,5$ $\therefore \sin(360° - 30°) = -\sin 30°$
$\cos 40°$ $= 0,8$	$\cos(180° - 40°)$ $= \cos 140°$ $= -0,8$ $\therefore \cos(180° - 40°) = -\cos 40°$	$\cos(180° + 40°)$ $= \cos 220°$ $= -0,8$ $\therefore \cos(180° + 40°) = -\cos 40°$	$\cos(360° - 40°)$ $= \cos 320°$ $= 0,8$ $\therefore \cos(360° - 40°) = \cos 40°$
$\tan 75°$ $= 3,7$	$\tan(180° - 75°)$ $= \tan 105°$ $= -3,7$ $\therefore \tan(180° - 75°) = -\tan 75°$	$\tan(180° + 75°)$ $= \tan 255°$ $= 3,7$ $\therefore \tan(180° + 75°) = \tan 75°$	$\tan(360° - 75°)$ $= \tan 285°$ $= -3,7$ $\therefore \tan(360° - 75°) = -\tan 75°$
$\sin 80°$ $= 0,98$	$\sin(180° - 80°)$ $= \sin 100°$ $= 0,98$ $\therefore \sin(180° - 80°) = \sin 80°$	$\sin(180° + 80°)$ $= \sin 260°$ $= -0,98$ $\therefore \sin(180° + 80°) = -\sin 80°$	$\sin(360° - 80°)$ $= \sin 280°$ $= -0,98$ $\therefore \sin(360° - 80°) = -\sin 80°$
All the angles of column 1 are angles that terminate in quadrant _1__.	All the angles of column 2 are angles that terminate in quadrant _2__.	All the angles of column 3 are angles that terminate in quadrant _3__.	All the angles of column 4 are angles that terminate in quadrant _4__.

(a) To reduce a trig ratio of an angle in Quad 2 you need to use $(180° - \text{acute})$
(b) To reduce a trig ratio of an angle in Quad 3 you need to use $(180° + \text{acute})$
(c) To reduce a trig ratio of an angle in Quad 4 you need to use $(360° - \text{acute})$
(d) **NB** The quadrant determines the sign $(+ \text{ or } -)$ of the trigonometric function.

Now you will be shown the reasons for the above results.

Function values of $180° - \theta$

In the diagram alongside, $OR = OR' = r$. If R has the coordinates $(x; y)$ then by a **reflection about the *y*-axis**, R' has the coordinates $(-x; y)$ and then OR' is the terminal arm of $180° - \theta$



$$\sin(180° - \theta) = \frac{y}{r} = \sin\theta$$

$$\cos(180° - \theta) = \frac{-x}{r} = -\cos\theta$$

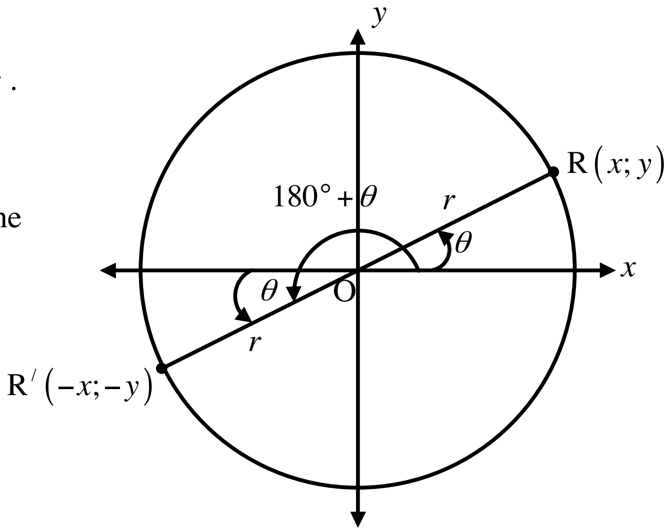
$$\tan(180° - \theta) = \frac{y}{-x} = -\tan\theta$$

$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$	$\tan(180° - \theta) = -\tan\theta$

It can be verified that for the above rules the angle θ can be any size.

Function values of $180° + \theta$

In the diagram alongside, $OR = OR' = r$. If R has the coordinates $(x; y)$ then by a **rotation of 180°**, R' has the coordinates $(-x; -y)$ and then OR' is the terminal arm of $180° + \theta$



$$\sin(180° + \theta) = \frac{-y}{r} = -\sin\theta$$

$$\cos(180° + \theta) = \frac{-x}{r} = -\cos\theta$$

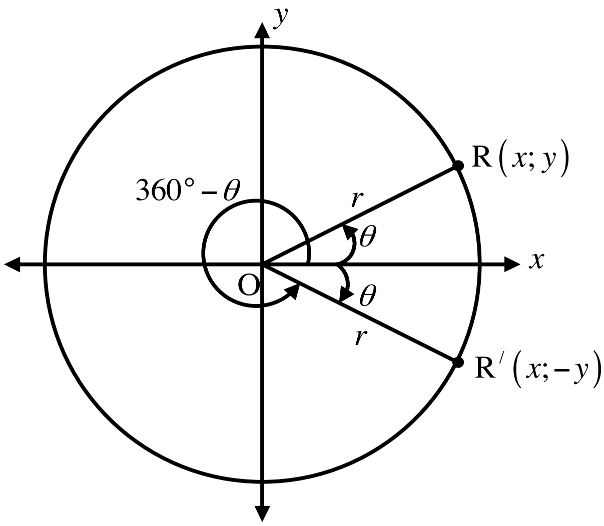
$$\tan(180° + \theta) = \frac{-y}{-x} = \frac{y}{x} = \tan\theta$$

$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$	$\tan(180° + \theta) = \tan\theta$

It can be verified that for the above rules the angle θ can be any size.

Function values of $360° - \theta$

In the diagram alongside, $OR = OR^/ = r$. If R has the coordinates $(x; y)$ then by a **reflection about the *x*-axis**, $R^/$ has the coordinates $(x; -y)$ and then $OR^/$ is the terminal arm of $360° - \theta$



$$\sin(360° - \theta) = \frac{-y}{r} = -\sin\theta$$

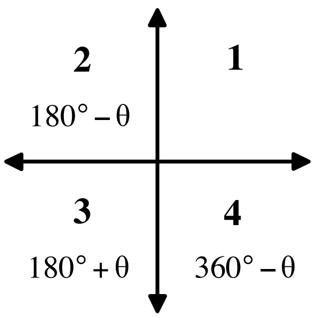
$$\cos(360° - \theta) = \frac{x}{r} = \cos\theta$$

$$\tan(360° - \theta) = \frac{-y}{x} = -\tan\theta$$

$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$	$\tan(360° - \theta) = -\tan\theta$

It can be verified that for the above rules the angle θ can be any size.

This can be summarised as follows.



If θ is an acute angle, the terminal arm of
$180° - \theta$ will lie in the **second** quadrant
$180° + \theta$ will lie in the **third** quadrant
$360° - \theta$ will lie in the **fourth** quadrant

EXAMPLE 1

Simplify the following as far as possible:

(a) $\dfrac{\cos(180° + \theta).}{\cos(180° - \theta)}$

3^{rd} quad (and cosine is negative in the 3^{rd})

$$\frac{\cos(180° + \theta)}{\cos(180° - \theta)} = \frac{-\cos\theta}{-\cos\theta} = 1$$

2^{nd} quad (and cosine is negative in the 2^{nd})

(b) $\dfrac{\tan(180^\circ+\theta).\sin(360^\circ-\theta)}{\tan(360^\circ-\theta)}$

$$\frac{\overset{3^{rd}}{\tan(180^\circ+\theta)}.\overset{4^{th}}{\sin(360^\circ-\theta)}}{\underset{4^{th}}{\tan(360^\circ-\theta)}}$$

Determine the quadrant and then the sign of the function value.

$= \dfrac{(+\tan\theta)(-\sin\theta)}{-\tan\theta}$

Cancel similar trigonometric functions of the **same angle**.

$= \sin\theta$

(c) $1-\tan^2(360^\circ-\theta)$

[It is important to remember that $\tan^2\theta = (\tan\theta)^2$. This holds for $\sin^2\theta$ and $\cos^2\theta$ as well]

$= 1-\left[\tan(360^\circ-\theta)\right]^2$

$= 1-(-\tan\theta)^2$

$= 1-(+\tan^2\theta)$

$= 1-\tan^2\theta$

Determine the quadrant and then the sign of the function value.

Take careful note of the above example which deals with the square of a trigonometric function.

EXERCISE 1

1. Write as function values of θ by referring to the diagram alongside. First add to the diagram which function is positive in which quadrant.

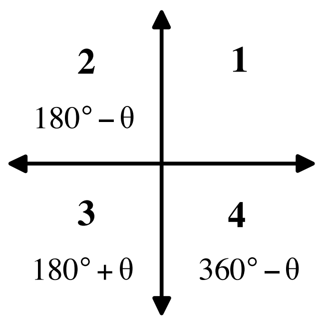
(a) $\sin(180^\circ+\theta)$ (b) $\sin(180^\circ-\theta)$

(c) $\cos(180^\circ-\theta)$ (d) $\cos(360^\circ-\theta)$

(e) $\tan(360^\circ-\theta)$ (f) $\tan(180^\circ+\theta)$

(g) $\cos^2(180^\circ+\theta)$ (h) $\tan^2(180^\circ-\theta)$

(i) $-\sin^2(360^\circ-\theta)$



2. Simplify the following as far as possible.
Take note of the following:

$\cos\theta+\cos\theta = 2\cos\theta$ and $\dfrac{\sin(180^\circ-\theta)}{\cos(180^\circ-\theta)} \neq \dfrac{\sin\cancel{(180^\circ-\theta)}}{\cos\cancel{(180^\circ-\theta)}} \neq \dfrac{\sin}{\cos}$.

(a) $\dfrac{\sin(180^\circ+\theta).\cos(180^\circ-\theta)}{\cos(180^\circ+\theta).\sin(180^\circ-\theta)}$ (b) $\tan^2\theta-\tan^2(180^\circ+\theta)$

(c) $\dfrac{\tan(360^\circ-\theta).\cos(180^\circ+\theta)}{\cos(360^\circ-\theta)-\cos(180^\circ+\theta)}$ (d) $\dfrac{\sin^2(360^\circ-\theta)}{\sin\theta.\sin(180^\circ+\theta)}$

Function values of $90° - \theta$

<u>Discovery exercise</u>

Use your calculator to calculate the following.

(a)	$\sin 10°$	and	$\cos 80°$
(b)	$\sin 20°$	and	$\cos 70°$
(c)	$\sin 30°$	and	$\cos 60°$
(d)	$\sin 40°$	and	$\cos 50°$

What do you notice about the answers of each pair?
What do you notice about the sum of the angles of each pair?

The answers of each pair above are equal and the sum of the angles in each case add up to 90°

$\sin 10° = \sin(90° - 80°)$ $\quad$ $\sin 20° = \sin(90° - 70°)$ $\quad$ $\sin 30° = \sin(90° - 60°)$

$\therefore \sin 10° = \cos 80°$ $\quad$ $\therefore \sin 20° = \cos 70°$ $\quad$ $\therefore \sin 30° = \cos 60°$

$\therefore \mathbf{\sin\theta = \cos(90° - \theta)}$

Or the other way around

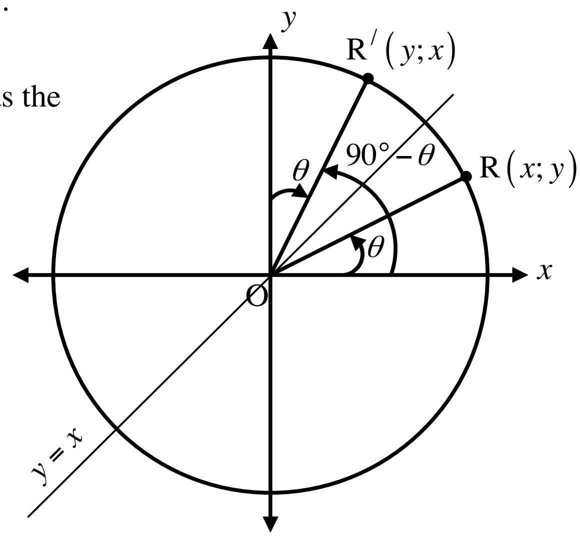
$\cos 80° = \cos(90° - 10°)$ $\quad$ $\cos 70° = \cos(90° - 20°)$ $\quad$ $\cos 60° = \cos(90° - 30°)$

$\therefore \cos 80° = \sin 10°$ $\quad$ $\therefore \cos 70° = \sin 20°$ $\quad$ $\therefore \cos 60° = \sin 30°$

$\therefore \mathbf{\cos\theta = \sin(90° - \theta)}$

Let us take a theoretical look at the function values of $90° - \theta$.

In the diagram alongside, $\text{OR} = \text{OR}' = r$.
If R has the coordinates $(x; y)$ then by a **reflection about the** $y = x$ **line**, R$'$ has the coordinates $(y; x)$ and then OR$'$ is the terminal arm of $90° - \theta$



$$\sin(90° - \theta) = \frac{x}{r} = \cos\theta$$

$$\cos(90° - \theta) = \frac{y}{r} = \sin\theta$$

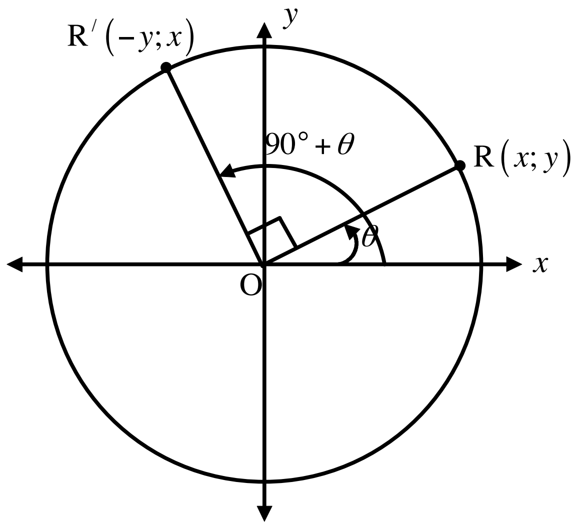
We will consider $\tan(90° - \theta)$ later on.

$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$

It can be verified that for the rules above the angle θ can be any size.

Function values of $90° + \theta$

In the diagram alongside, $OR = OR' = r$.
If R has the coordinates $(y;x)$ then by
a **rotation of 90° anti-clockwise**, R' has the coordinates $(-y;x)$ and then OR' is the terminal arm of $90° + \theta$



$$\sin(90° + \theta) = \frac{x}{r} = \cos\theta$$

$$\cos(90° + \theta) = \frac{-y}{r} = -\sin\theta$$

We will consider $\tan(90° + \theta)$ later on.

It is important to note that the terminal arm of $90° + \theta$ lies in the 2nd quadrant if θ is acute

$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$

It can be verified that for the rules above the angle θ can be any size.

Function values of $90 \pm \theta$ are the only function values that change the trigonometric function you are working with (when simplifying). When you are dealing with function values of $90 \pm \theta$, ***sine*** **changes to** ***cosine*** **and** ***cosine*** **changes to** ***sine*.**

EXAMPLE 2

Simplify the following as far as possible:

(a) $\dfrac{\cos(90° + \theta)}{\sin(360° - \theta)}$

[Note that: $360° - \theta$ lies in 4th quad (sine is negative in the 4th)
and $\cos(90° + \theta) = -\sin\theta$]

$= \dfrac{-\sin\theta}{-\sin\theta}$
$= 1$

(b) $\dfrac{\sin(360°-\theta).\sin(90°+\theta)}{\cos(90°-\theta).\cos(360°-\theta)}$

$$\frac{\overset{4^{th}}{\sin(360°-\theta)}.\overset{2^{nd}}{\sin(90°+\theta)}}{\underset{1^{st}}{\cos(90°-\theta)}.\underset{4^{th}}{\cos(360°-\theta)}}$$

Determine the quadrants

$= \dfrac{(-\sin\theta).\cos\theta}{\sin\theta.\cos\theta}$

$= -1$

Determine the sign and then the trig function.

*(c) Show that $\sin(270°-\beta) = -\cos\beta$

Consider the fact that $270° = 180° + 90°$

$\therefore \sin(270°-\beta)$

$= \sin(180° + 90° - \beta)$

$90° - \beta = \theta$

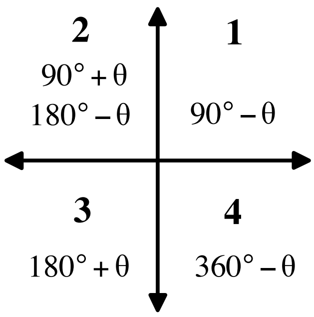
$= \sin(180° + (90° - \beta))$ Applying $\sin(180° + (\theta)) = -\sin(\theta)$

$= -\sin(90° - \beta)$

$= -\cos\beta$ $\sin(90° - \beta) = \cos\beta$

EXERCISE 2

1. Write the following as function values of θ without referring to your textbook. Use the diagram alongside to assist you.



(a)	$\sin(90°-\theta)$	(b)	$\sin(180°-\theta)$
(c)	$\cos(90°+\theta)$	(d)	$\cos(180°+\theta)$
(e)	$\cos(360°-\theta)$	(f)	$\sin(90°+\theta)$
(g)	$\sin(180°+\theta)$	(h)	$\cos(90°-\theta)$

2. Simplify the following:

(a) $\dfrac{\cos(180°+\theta).\cos(90°-\theta)}{\sin(90°+\theta).\sin(180°-\theta)}$ (b) $\sin(90°-\theta).\cos(180°+\theta)$

(c) $\tan(180°+\theta).\cos(90°+\theta)+\sin(360°-\theta).\tan(180°-\theta)$

3. Simplify the following:

(a) $\sin^2(180°+\theta)-\cos^2(90°-\theta)$ (b) $\dfrac{\sin^2(360°-\theta)}{\cos(90°+\theta).\sin(180°-\theta)}$

*(c) $\cos(270°-\theta)$ *(d) $\sin(270°+\theta)$

*(e) $\cos(270°+\theta)$

Numerical examples of angles between 0° and 360°

EXAMPLE 3

Write the following as the function value of an acute angle.

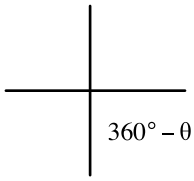
(a) $\sin 290°$ (b) $\tan 197°$ (c) $\cos 120°$

Solutions

(a) $\sin 290°$

$= \sin(360° - 70°)$ 290° lies in the 4^{th} quad

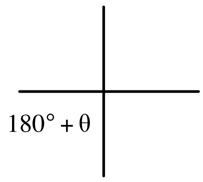
$= -\sin 70°$ sine is **neg.** in the 4^{th} quad



(b) $\tan 197°$

$= \tan(180° + 17°)$ 197° lies in the 3^{rd} quad

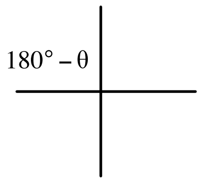
$= +\tan 17°$ tan is **pos.** in the 3^{rd} quad



(c) $\cos 120°$

$= \cos(180° - 60°)$ 120° lies in the 2^{nd} quad

$= -\cos 60°$ cosine is **neg.** in the 2^{nd} quad



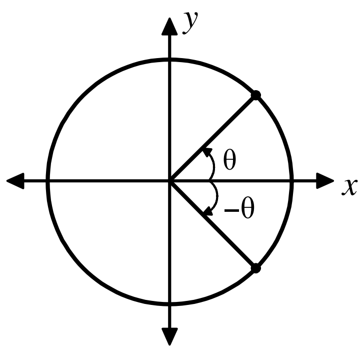
EXERCISE 3

Write the following as function values of acute angles.

(a)	$\sin 310°$	(b)	$\tan 110°$	(c)	$\cos 95°$
(d)	$\tan 235°$	(e)	$\sin 250°$	(f)	$\cos 222°$
(g)	$\cos 335°$	(h)	$\sin 120°$	(i)	$\cos 304°$

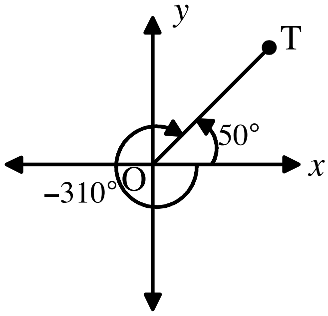
Function values of $-\theta$ (negative angles)

It is important to note that on the Cartesian plane an angle is considered to be **positive** if the rotation is in an **anti-clockwise** direction and **negative** if the rotation is **clockwise**.



Consider the following diagram.

The anti-clockwise rotation of 50° (positive) and the clockwise rotation of 310° (negative) have the same terminal arm in the first quadrant.

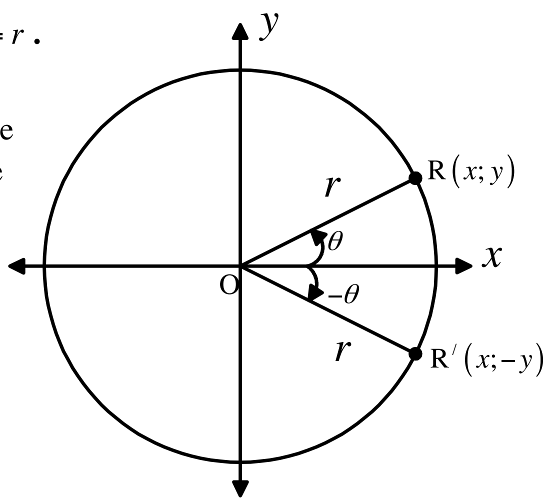


In the diagram alongside, $OR = OR^{/} = r$ **.**
If R has the coordinates $(x; y)$ then by
a **reflection about the *x*-axis**, $R^{/}$ has the coordinates $(x; -y)$ and then $OR^{/}$ is the terminal arm of $-\theta$

$$\sin(-\theta) = \frac{-y}{r} = -\sin\theta$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta$$

$$\tan(-\theta) = \frac{-y}{x} = -\tan\theta$$

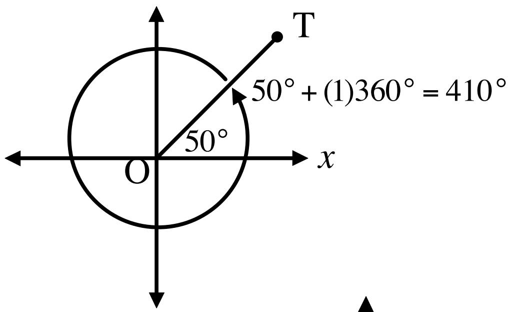


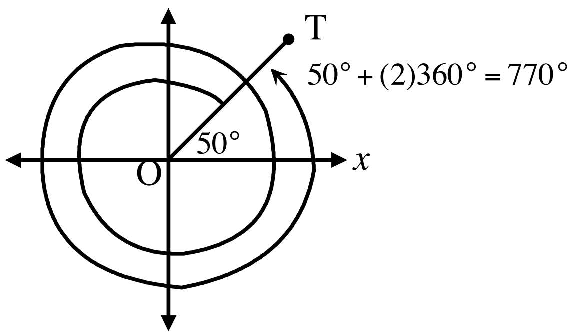
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$

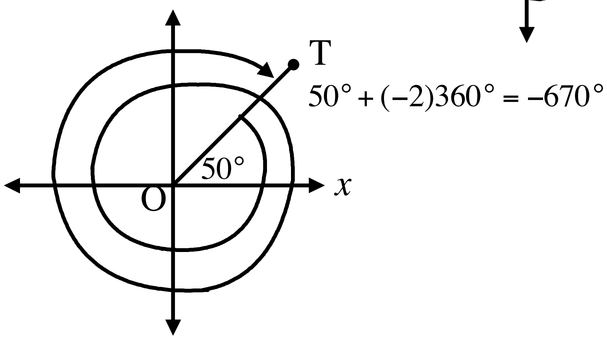
It can be verified that for these rules the angle θ can be any size.

Function values of $\theta + k.360°$

Consider, for example, the angles $410°$, $770°$ and $-670°$.







You will notice that we can rotate $360°$ from $50°$ in a clockwise or anti-clockwise direction any number of times.
We can say that the angles $50° + (k)360°$ where $k \in \mathbb{Z}$ have the same terminal arm. In general, the angles $\theta + (k)360°$ where $k \in \mathbb{Z}$ have the same terminal arm.

Note: $\sin 410° = 0{,}766$ and $\sin 50° = 0{,}766$

$\therefore \sin(50° + (1)360°) = \sin 50°$ $\quad$ $(410° = 50° + (1)360°)$

$\sin 770° = 0{,}766$ and $\sin 50° = 0{,}766$

$\therefore \sin(50° + (2)360°) = \sin 50°$ $(770° = 50° + (2)360°)$
$\sin(-670°) = 0{,}766$ and $\sin 50° = 0{,}766$
$\therefore \sin(50° + (-2)360°) = \sin 50°$ $(-670° = 50° + (-2)360°)$

So, in general, the following reduction rules apply:

Take note:
$k.360°$ refers to multiples of 360° either being added or subtracted.

$\sin(\theta + k.360°) = \sin\theta$	or	$\sin(k.360° + \theta) = \sin\theta,\ k \in Z$
$\cos(\theta + k.360°) = \cos\theta$	or	$\cos(k.360° + \theta) = \cos\theta,\ k \in Z$
$\tan(\theta + k.360°) = \tan\theta$	or	$\tan(k.360° + \theta) = \tan\theta,\ k \in Z$

EXAMPLE 4

Simplify the following to a function value of θ:

(1) $\sin(360° + \theta)$ (2) $\cos(540° + \beta)$
(3) $\tan(\theta - 180°)$ (4) $\sin(-\theta - 90°)$

Solutions

(1) $\sin(360° + \theta)$
$= \sin\big((1)360° + \theta\big)$
$= \sin\theta$

(2) $\cos(540° + \beta)$
$= \cos(360° + 180° + \beta)$ $540° = 360° + 180°$
$= \cos\big(360° + (180° + \beta)\big)$
$= \cos(180° + \beta)$ $\cos\big((1)360 + \theta\big) = \cos\theta$
$= -\cos\theta$

(3) $\tan(\theta - 180°)$
$= \tan\big(-(\theta + 180°)\big)$ Take out a negative sign
$= -\tan(\theta + 180°)$ $\tan(-angle) = -\tan(angle)$
$= -\tan(180° + \theta)$
$= -\tan\theta$

(4) $\sin(-\theta - 90°)$
$= \sin\big(-(\theta + 90°)\big)$ Take out a negative sign
$= -\sin(\theta + 90°)$ $\sin(-angle) = -\sin(angle)$
$= -\sin(90° + \theta)$
$= -\cos\theta$ $\sin(90° + \theta) = \cos\theta$

Numerical examples of negative angles and angles greater than 360°

EXAMPLE 5

Write the following as a function value of a positive acute angle.

(a) $\sin(-30°)$ (b) $\cos(-40°)$

(c) $\tan(-150°)$ (d) $\cos 920°$

Solutions

(a) $\sin(-30°)$
$= -\sin 30° \ldots \quad \sin(-\theta) = -\sin\theta$

(b) $\cos(-40°)$
$= \cos 40° \ldots \quad \cos(-\theta) = \cos\theta$

(c) $\tan(-150°)$

$= -\tan 150°$ Apply the rule $\tan(-\theta) = \tan\theta$

$= -\tan(180° - 30°)$ 150° lies in the 2nd quad.

$= -(-\tan 30°)$ tan is **neg.** in the 2nd quad

$= \tan 30°$

(d) $\cos 920°$

$= \cos(200° + (2)360°)$

$= \cos 200°$

$= \cos(180° + 20°)$ 200° lies in the 3rd quad

$= -\cos 20°$ cosine is **neg.** in the 3rd quad

Notice:
Because 920° is greater than 360°, you may subtract 360° twice in order to work with the trig ratio of an angle in the interval $[0°; 360°]$.
In other words, you can proceed as follows:
$\cos 920° = \cos 200°$ (subtract 360° twice)

EXERCISE 4

1. Write each of the following as a function value of θ.

(a) $\tan(-\theta)$ (b) $\cos(-\theta)$ (c) $\sin(-\theta)$

(d) $\sin(\theta - 180°)$ (e) $\tan(-\theta - 180°)$ (f) $\cos(\theta - 90°)$

(g) $\cos(360° + \theta)$ (h) $\sin(\theta + 90°)$ (i) $\tan(720° - \theta)$

(j) $\cos(540° - \theta)$ (k) $\sin(-\theta - 90°)$ (l) $\cos(\theta - 270°)$

2. Simplify the following:

(a) $\dfrac{\cos(\theta - 180°).\cos(90° - \theta)}{\sin(90° + \theta).\sin(-\theta - 180°)}$ (b) $\dfrac{\cos(\theta - 90°).\tan(-\theta)}{\sin(-\theta).\tan(720° - \theta)}$

(c) $\dfrac{\sin(\beta - 180°)\tan(-\beta - 180°).\cos(180° + \beta)}{\cos(-\beta).\sin(360° + \beta)}$

3. Write each of the following as a function value of a positive acute angle.

(a)	$\tan(-50°)$	(b)	$\cos(-240°)$	(c)	$\sin(-20°)$
(d)	$\tan(-210°)$	(e)	$\cos(-135°)$	(f)	$\sin(-125°)$
(g)	$\tan 650°$	(h)	$\cos 765°$	(i)	$\sin 485°$
(j)	$\sin(-415°)$				

<u>EXAMPLE 6</u>

If $\cos 20° = t$ determine the value(s) of the following in terms of t.

(a)	$\cos 160°$	(b)	$\cos(-200°)$	(c)	$\sin 250°$
(d)	$\sin 20°$	(e)	$\tan 20°$		

<u>Solutions</u>

(a) to (c) are examples of which we will make use of reduction formulae only.

(a) $\cos 160°$

$= \cos(180° - 20°)$ 2nd quad

$= -\cos 20°$

$= -t$

(b) $\cos(-200°)$

$= \cos 200°$ Apply $\cos(-\theta) = \cos\theta$

$= \cos(180° + 20°)$ 3rd quad

$= -\cos 20° = -t$

(c) $\sin 250°$

$= \sin(180° + 70°)$ 3rd quad

$= -\sin 70°$

Notice that the acute angle is different (70° and not 20°)

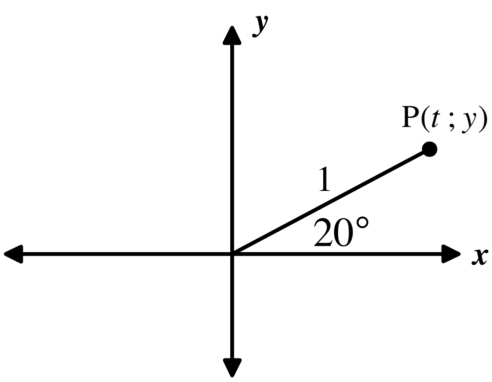
Now continue as follows:

$\therefore -\sin 70°$

$= -\cos 20°$ $\sin 70° = \sin(90° - 20°) = \cos 20°$

$= -t$

The functions in (d) and (e) require you to draw a diagram.



$\cos 20° = t = \frac{t}{1} = \frac{x}{r}$

$\therefore y^2 = 1^2 - t^2$

$\therefore y = \sqrt{1 - t^2}$

(d) $\sin 20° = \frac{y}{r} = \frac{\sqrt{1-t^2}}{1} = \sqrt{1-t^2}$

(e) $\tan 20° = \frac{y}{x} = \frac{\sqrt{1-t^2}}{t}$

EXERCISE 5

1. If $\sin 40° = p$ write the following in terms of p.

(a)	$\sin 140°$	(b)	$\sin 220°$	(c)	$\sin(-140°)$
(d)	$\cos 50°$	(e)	$\sin 50°$	(f)	$\tan 220°$

2. If $\cos 35° = k$ write the following in terms of k.

(a)	$\cos 215°$	(b)	$\sin 55°$	(c)	$\cos(-35°)$
(d)	$\sin 665°$	(e)	$\sin 325°$		

3. If $\tan 22° = t$ write the following in terms of t.

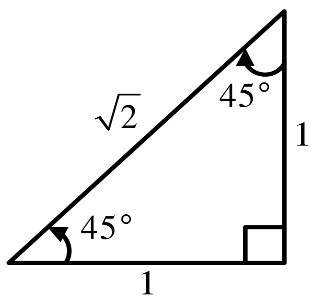
(a)	$\tan 202°$	(b)	$\tan 338°$	(c)	$\tan(-22°)$
(d)	$\tan 518°$	(e)	$\cos 22°$		

Special angles $(30°, 45°, 60°)$ and angles $(0°, 90°, 180°, 270°, 360°)$

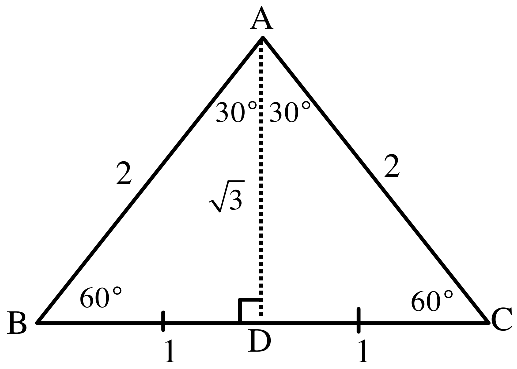
Revision of Grade 10 work

Consider the following two triangles for special angles $30°$, $45°$ and $60°$.

Triangle A



Triangle B



From triangle A we know:

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{1}{1} = 1$$

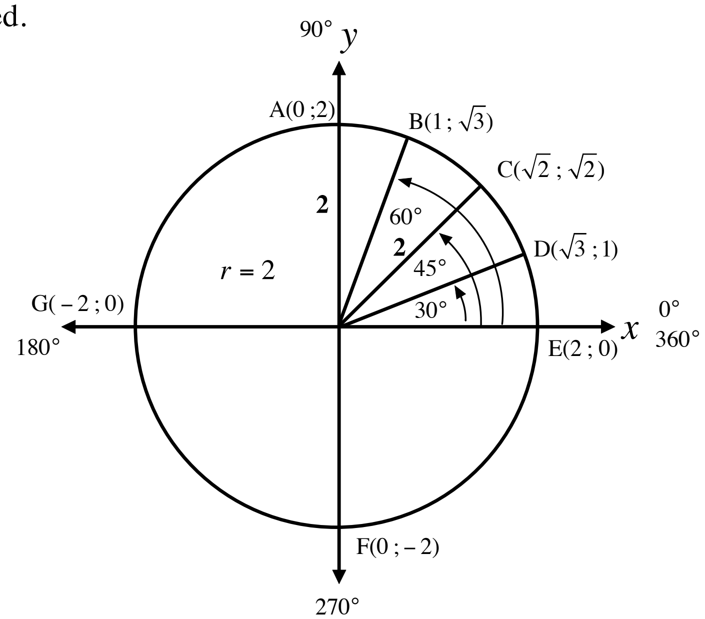
From triangle B we know:

$\sin 30° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$ and $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

For the angles $0°$, $90°$, $180°$, $270°$, $360°$ and $30°$, $45°$, $60°$ the diagram below can also be used.



EXAMPLE 7

Use the special angle diagrams to calculate the following. Verify your answers by using a calculator.

(a)	$\sin 60°$	(b)	$\tan 45°$
(c)	$\cos 90°$	(d)	$\sin 270°$

Solutions

(a) $\sin 60° = \dfrac{y}{r} = \dfrac{\sqrt{3}}{2}$ (b) $\tan 45° = \dfrac{y}{x} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$

(c) $\cos 90° = \dfrac{x}{r} = \dfrac{0}{2} = 0$ (d) $\sin 270° = \dfrac{y}{r} = \dfrac{-2}{2} = -1$

EXAMPLE 8

Evaluate the following without the use of a calculator. You may use your calculator to verify each step and your final answer.

(a)	$\cos^2 150°$	(b)	$\tan(-300°).\sin 600°$
(c)	$\dfrac{\cos(-200°)}{\sin 290°}$	(d)	$\dfrac{\cos 180°.\sin 225°.\cos 80°}{\sin 170°.\tan 135°}$

Solutions

(a) $\cos^2 150°$ 2^{nd}

$= [\cos(180° - 30°)]^2$ cosine is negative in the 2^{nd}

$= (-\cos 30°)^2$

$= \left(-\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{3}{4}$

(b) $\tan(-300°).\sin 600°$ — Note that $600°$ is larger than $360°$ and apply $\tan(-\theta) = -\tan\theta$

$= (-\tan 300°).\sin(240° + (1)360°)$

$= (-\tan 300°).\sin 240°$

$= \left[-\tan\left(360° - 60°\right)\right].\sin\left(180° + 60°\right)$ — Use the applicable reduction formulae to reduce to an acute angle

$= \left[-(-\tan 60°)\right].(-\sin 60°)$

$= (\tan 60°).(-\sin 60°)$

$= \left(\frac{\sqrt{3}}{1}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3}{2} = -1\frac{1}{2}$

(c) $\frac{\cos(-200°)}{\sin 290°}$

$= \frac{\cos 200°}{\sin(360° - 70°)}$ — $\cos(-\theta) = \cos\theta$

$= \frac{\cos(180° + 20°)}{-\sin 70°}$ — $\sin 70° = \sin(90° - 20°) = \cos 20°$

$= \frac{-\cos 20°}{-\cos 20°} = 1$ — NB to work with the same angle

(d) $\frac{\cos 180°.\sin 225°.\cos 80°}{\sin 170°.\tan 135°}$

$= \frac{\left(\frac{-2}{2}\right).\sin\left(180° + 45°\right).\cos 80°}{\sin\left(180° - 10°\right).\tan\left(180° - 45°\right)}$

$= \frac{\left(-1\right).\left(-\sin 45°\right).\cos 80°}{(\sin 10°).(-\tan 45°)}$

$= \frac{\left(-1\right).\left(-\frac{\sqrt{2}}{2}\right).\sin 10°}{(\sin 10°).(-1)}$ — $\cos 80° = \cos\left(90° - 10°\right) = \sin 10°$

Bear this in mind when the acute angle is not special.

$= -\frac{\sqrt{2}}{2}$

Note:
When you are expected to simplify or calculate function values of numerical angles then apply the following rules:

1. Use reduction formulae to make the angles acute.
2. Identify the special angles and then substitute the function value.
3. For those **angles which are not special,** use the reduction formulae $90 - \theta$ so as to work with function values of the **same angle.**

EXERCISE 6

1. Evaluate, without using a calculator.

(a) $\cos 330°.\sin 60°$

(b) $\dfrac{\sin 135°}{\cos 225°}$

(c) $\tan 315° - 2\cos 60° + \sin 210°$

(d) $\sin 570° + \cos 240° - \tan 135°$

(e) $\dfrac{\tan 330°}{\sin 330°}$

(f) $\dfrac{\sin 410°}{\cos 40°}$

(g) $\sin 150° - \tan 240°.\cos 210°$

(h) $\tan 120°.\cos 210° - \sin^2 315°$

(i) $\dfrac{\tan 150°}{\tan 240°} - \dfrac{\sin 300°}{\sin 120°}$

(j) $\dfrac{\tan 315° - \cos 1020°}{\sin 150° + \tan(-135°)}$

(k) $\dfrac{\tan 225° - \sin x}{\sin x + \sin 270°}$

(l) $\cos(-315°).\sin 315° - \dfrac{\cos 20°}{\sin 250°}$

2. Prove without using a calculator: $\dfrac{\cos 315° + 1}{\sin 315° - 1} = -1$

3. Evaluate without using a calculator: $\sqrt{4^{\sin 150°}.2^{3\tan 225°}}$

4. Show that $\theta = 30°$ is a solution to $(\sin\theta)^{\sin\theta} = \dfrac{1}{\sqrt{2}}$

TRIGONOMETRIC IDENTITIES

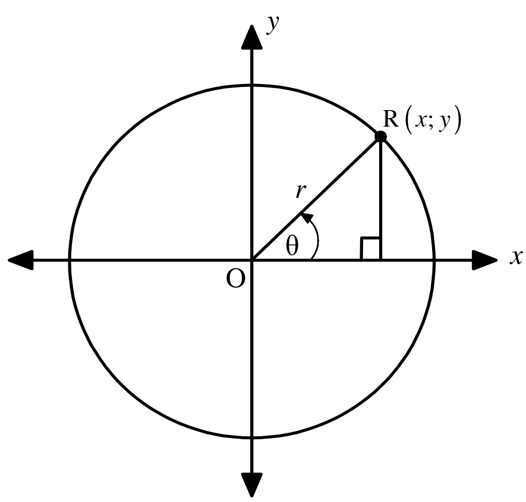
An identity is a mathematical statement that is true for all values of the variable excluding the values for which the statement is not defined.

Two very important identities will now be considered and then proved.

Quotient identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\theta \neq 90° + k.180°$

Square identity: $\sin^2\theta + \cos^2\theta = 1$

$R(x; y)$ is any point on the terminal arm of θ in the standard position.



Identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	**Identity:** $\sin^2\theta + \cos^2\theta = 1$
Proof: Refer to the diagram.	**Proof:** Refer to the diagram.
Consider LHS:	Consider LHS:
$\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{y}{r}}{\frac{x}{r}}$	$\sin^2\theta + \cos^2\theta$
$= \dfrac{y}{r} \times \dfrac{r}{x}$	$= \dfrac{y^2}{r^2} + \dfrac{x^2}{r^2}$
$= \dfrac{y}{x}$	$= \dfrac{x^2 + y^2}{r^2}$ but $x^2 + y^2 = r^2$ ....Pythagoras
$= \tan\theta = \text{RHS}$	$= \dfrac{r^2}{r^2}$
	$= 1 = \text{RHS}$
$\therefore \tan\theta = \dfrac{\sin\theta}{\cos\theta},\ \theta \neq 90° + k.180°$	$\therefore \sin^2\theta + \cos^2\theta = 1$ for all values of θ

Summary of the trigonometric identities

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta},\ \theta \neq 90° + k.180°$
2. $\sin^2\theta + \cos^2\theta = 1$ for all values of θ.
   This identity can be written in the following ways:
   (a) $\cos^2\theta = 1 - \sin^2\theta$ or (b) $\sin^2\theta = 1 - \cos^2\theta$

** Factorising and working with fractions is a crucial skill to have mastered before attempting the following types of questions.**

<u>EXAMPLE 9</u>

(a) Factorise the following:

(1) $2\sin\theta\cos\theta - \cos\theta$ (2) $2\cos^2\theta - 3\cos\theta - 2$

(3) $\sin^2\theta + 2\sin\theta.\cos\theta + \cos^2\theta$ (4) $1 - \cos^2\theta$

(b) Show that $1 + \sin\theta - \cos^2\theta = \sin\theta(1 + \sin\theta)$

(c) Use your trigonometric identities to simplify the following to a single trigonometric function.

(1) $(1 - \sin\theta)(1 + \sin\theta)$ (2) $\cos^2\theta - 1$

(3) $\tan^2\theta - \tan^2\theta.\sin^2\theta$ (4) $(\cos\theta + \sin\theta)^2$

Solutions

(a) (1) $2\sin\theta\cos\theta - \cos\theta$

$= \cos\theta(2\sin\theta - 1)$

($\cos\theta$ is the highest common factor)

(2) $2\cos^2\theta - 3\cos\theta - 2$

This is a trigonometric trinomial:

$= (2\cos\theta + 1)(\cos\theta - 2)$

An alternative method can be used by introducing another letter so as to simplify the factorising process:

Let $\cos\theta = k$

$\therefore 2\cos^2\theta - 3\cos\theta - 2$

$= 2k^2 - 3k - 2$

$= (2k + 1)(k - 2)$

$= (2\cos\theta + 1)(\cos\theta - 2)$

(3) $\sin^2\theta - 2\sin\theta.\cos\theta + \cos^2\theta$

This is also a trigonometric trinomial:

$\therefore (\sin\theta - \cos\theta)(\sin\theta - \cos\theta)$

Alternatively: Let $\sin\theta = k$ and $\cos\theta = p$

$\therefore \sin^2\theta - 2\sin\theta.\cos\theta + \cos^2\theta$

$= k^2 - 2kp + p^2$

$= (k - p)(k - p)$

$= (\sin\theta - \cos\theta)(\sin\theta - \cos\theta)$

(4) $1 - \cos^2\theta$

$= (1 - \cos\theta)(1 + \cos\theta)$ (Difference of two squares)

(b) Show that $1 + \sin\theta - \cos^2\theta = \sin\theta(1 + \sin\theta)$

LHS: $1 + \sin\theta - \cos^2\theta$

It is clear that we should be working with the same trigonometric function, which in this case will be sine. Therefore continue as follows.

$= 1 + \sin\theta - (1 - \sin^2\theta)$ Substitute $\cos^2\theta = 1 - \sin^2\theta$

$= 1 + \sin\theta - 1 + \sin^2\theta$

$= \sin\theta + \sin^2\theta$

$= \sin\theta(1 + \sin\theta) = \text{RHS}$

(c)(i) $(1 - \sin\theta)(1 + \sin\theta)$

$= 1 - \sin^2\theta$

$= \cos^2\theta$ (Use the identity $1 - \sin^2\theta = \cos^2\theta$)

(c)(ii) $\cos^2\theta - 1 = -\sin^2\theta$ (By manipulating $\sin^2\theta + \cos^2\theta = 1$ we can deduce this) OR

$\cos^2\theta - 1 = -(1-\cos^2\theta)$ Factorise (or Change of sign rule)

$= -\sin^2\theta$ Use the identity: $\sin^2\theta = 1 - \cos^2\theta$

(c)(iii) $\tan^2\theta - \tan^2\theta.\sin^2\theta$

$= \tan^2\theta(1-\sin^2\theta)$ Factorise by taking out the HCF

$= \dfrac{\sin^2\theta}{\cos^2\theta}(\cos^2\theta)$ If $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ then $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}\left(\dfrac{\cos^2\theta}{1}\right)$

$= \sin^2\theta$

(c)(iv) $(\cos\theta + \sin\theta)^2$ Reminder: $(\cos\theta+\sin\theta)^2 = (\cos\theta+\sin\theta)(\cos\theta+\sin\theta)$

$= \cos^2\theta + 2\cos\theta.\sin\theta + \sin^2\theta$ or use the short method for squaring

$= \cos^2\theta + \sin^2\theta + 2\cos\theta.\sin\theta$ a binomial.

$= 1 + 2\cos\theta.\sin\theta$ $\cos^2\theta + \sin^2\theta = 1$

EXERCISE 7

Simplify the following as far as possible.

(a) $\dfrac{\sin\theta}{\cos\theta}$ (b) $\cos\theta.\tan\theta$ (c) $\dfrac{\sin\theta}{\tan\theta}$

(d) $1+\tan^2\theta$ (e) $\dfrac{1-\cos^2\theta}{\sin^2 - 1}$ (f) $\cos^4\theta + \cos^2\theta.\sin^2\theta$

Proving of identities

When proving identities it is important to remember the saying "Innocent until proven guilty". It means that you are not allowed to work with the "$=$" sign, because the statement is not true yet. You have to prove that it is equal. Therefore you have to split it up and work with the LHS and RHS separately and conclude with LHS = RHS.

EXAMPLE 10

(a) Prove that: $\dfrac{1}{\cos^2\theta} - 1 = \tan^2\theta$

LHS: $\dfrac{1}{\cos^2\theta} - 1$ Work with the left hand side (LHS)

$= \dfrac{1-1(\cos^2\theta)}{\cos^2\theta}$ Fractions $\Rightarrow$ Find LCD and simplify

$= \dfrac{1-\cos^2\theta}{\cos^2\theta}$ Use the identity: $1-\cos^2\theta = \sin^2\theta$

$= \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = \text{RHS}$

(b) Prove that: $\tan\theta + \dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1}{\cos\theta}$

LHS: $\tan\theta + \dfrac{\cos\theta}{1+\sin\theta}$	Work with the LHS
$= \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$	Use the identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
$= \dfrac{\sin\theta(1+\sin\theta)+\cos\theta(\cos\theta)}{\cos\theta(1+\sin\theta)}$	Fractions $\Rightarrow$ Find LCD and simplify
$= \dfrac{\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1+\sin\theta)}$	Use the identity: $\sin^2\theta+\cos^2\theta=1$
$= \dfrac{\sin\theta+1}{\cos\theta(1+\sin\theta)}$	
$= \dfrac{1}{\cos\theta} = \text{RHS}$	

Guidelines you can use for the proving of identities

- Split LHS and RHS
- Use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- Fractions $\Rightarrow$ Find LCD and simplify
- Factorise or use the identity $\sin^2\theta+\cos^2\theta=1$ (in all its forms) or use both.

EXERCISE 8

1. Prove that the following statements are identities.

(a) $(\sin\theta+\cos\theta)^2 = 1+2\sin\theta.\cos\theta$ (An important result to remember)

(b) $\dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$ (An important result to remember)

(c) $\dfrac{1}{\sin^2\theta} - \dfrac{\cos^2\theta}{\sin^2\theta} = 1$

(d) $\dfrac{\tan x.\cos x}{\sin x} = 1$

(e) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

(f) $\sin^2\theta + \sin^2\theta.\tan^2\theta = \tan^2\theta$

(g) $\cos\theta(1+\tan\theta) = \cos\theta+\sin\theta$

(h) $\dfrac{1-\cos^2\theta}{\cos^2\theta+2\cos\theta+1} = \dfrac{1-\cos\theta}{1+\cos\theta}$

(i) $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta} = \dfrac{2}{\sin^2\theta}$

(j) $\dfrac{1}{\sin\theta} + \dfrac{1}{\tan\theta} = \dfrac{1+\cos\theta}{\sin(180°-\theta)}$

2. In order to prove the following, manipulate both the LHS and the RHS:

(a) $\dfrac{1}{1-\sin x}-\dfrac{1}{1+\sin x}=\dfrac{2\tan x}{\cos x}$ (b) $\dfrac{1+\sin x}{1-\sin x}-\dfrac{1-\sin x}{1+\sin x}=\dfrac{4\tan x}{\cos x}$

*3. Prove that:

(a) $\dfrac{1+\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{2}{\sin\theta}$ (b) $\left(\dfrac{1}{\cos\theta}-\tan\theta\right)^2=\dfrac{1-\sin\theta}{1+\sin\theta}$

(c) $\dfrac{1+2\sin\theta.\cos\theta}{\sin\theta+\cos\theta}=\sin\theta+\cos\theta$ (Consider 1(a))

Further examples involving identities and reduction formulae

EXAMPLE 11

Simplify as far as possible by using **reduction formulae and identities**.

(a) $\dfrac{\tan 225°+\cos(180°-\theta).\sin(90°+\theta)}{\sin(-\theta)}$

(b) $\cos^2 140°-\tan^2 40°.\cos 220°.\sin 50°$

Solutions

(a) $\dfrac{\tan 225°+\cos(180°-\theta).\sin(90°+\theta)}{\sin(-\theta)}$ $\qquad \tan 225°=\tan(180°+45°)$

$=\dfrac{\tan 45°+(-\cos\theta).(\cos\theta)}{-\sin\theta}$ Identify quadrants and then the signs

$=\dfrac{1-\cos^2\theta}{-\sin\theta}$ 45° is a special angle

$=\dfrac{\sin^2\theta}{-\sin\theta}=-\sin\theta$

(b) $\cos^2 140°-\tan^2 40°.\cos 220°.\sin 50°$

The first step will be to reduce all angles to acute angles using the applicable reduction formulae.

$=\left[\cos(180°-40°)\right]^2-\tan^2 40°.\cos(180°+40°).\sin 50°$

$=(-\cos 40°)^2-\tan^2 40°.(-\cos 40°).\sin 50°$ (50° is the odd one out)

Secondly, if all of the angles are acute then one should attempt to work with the same angle. $(\sin 50°=\sin(90°-40°)=\cos 40°)$

$=\cos^2 40°+\tan^2 40°.\cos 40°.\cos 40°$

$=\cos^2 40°+\dfrac{\sin^2 40°}{\cos^2 40°}.\dfrac{\cos^2 40°}{1}$

$=\cos^2 40+\sin^2 40$ $\qquad (\sin^2\theta+\cos^2\theta=1)$

$=1$

EXERCISE 9

Simplify as far as possible by using **reduction formulae and identities**.
Calculators may not be used in this question.

(a) $\dfrac{\cos(90°-\theta)}{\sin(180°-\theta)}-\sin^2(-\theta)$ (b) $\dfrac{\cos(180°+\theta).\tan(180°-\theta)}{\sin(-\theta)}$

(c) $\dfrac{\tan(180°+\theta)\cos(360°+\theta)}{\sin(180°+\theta).\cos(90°+\theta)+\cos^2(360°-\theta)}$

(d) $\cos^2 112°+\sin^2 68°$ (e) $\cos^2 260°-\sin 100°.\sin 280°$

(f) $\sin 70°.\cos 200°-\cos^2(-70°)$

TRIGONOMETRIC EQUATIONS

Before dealing with this topic, it is absolutely essential for you to ensure that your calculator is on "degrees".

Find the numerical value of the following:

$\sin 30° =$ ______ and $\sin 150° =$ _____

$\therefore$ The function value of two **different angles** gives the *same numerical value*.

$\cos 55° =$ ______ and $\cos 305° =$ _____

$\therefore$ The function value of two **different angles** gives the *same numerical value*.

$\tan 45° =$ ______ and $\tan 225° =$ _____

$\therefore$ The function value of two **different angles** gives the *same numerical value*.

Find the angle if the numerical value is given.

$\sin\theta = \dfrac{1}{2}$	$\therefore \theta = \sin^{-1}\left(\dfrac{1}{2}\right) = 30°$	but what about 150°?
$\cos\theta = 0{,}5736$	$\therefore \theta = \cos^{-1}(0{,}5736) = 55°$	but what about 305°?
$\tan\theta = 1$	$\therefore \theta = \tan^{-1}(1) = 45°$	but what about 225°?

Thus it is important to note that the function value of more than one angle can generate a specific numerical value.

When solving trigonometric equations you will have to consider when a specific trigonometric function is positive or negative as that will determine in which quadrant the angle will lie.

When working in a specific quadrant we will use the applicable formula:

$(180° -$ acute reference angle$)$	for any angle in Quadrant 2
$(180° +$ acute reference angle$)$	for any angle in Quadrant 3
$(360° -$ acute reference angle$)$	for any angle in Quadrant 4

EXAMPLE 12

(a) Solve for θ if $\sin\theta = \frac{1}{2}$, $\theta \in (0°;360°)$

Reference angle = $\sin^{-1}\left(\frac{1}{2}\right) = 30°$ (Reference angle is the **acute** angle that will generate the **positive numerical value**)

$\sin\theta$ is equal to $+\frac{1}{2}$ and sine is positive in Quadrants 1 and 2

Quadrant 1		Quadrant 2
$\theta = 30°$	or	$\theta = 180° - \text{acute reference angle}$
		$\therefore \theta = 180° - 30° = 150°$

(b) Solve for θ if $\cos\theta = -\frac{5}{7}$, $\theta \in (0°;360°)$, rounded off to two decimal places.

Reference angle = $\cos^{-1}\left(\oplus\frac{5}{7}\right) = 44{,}4153086°$

(The reference angle is the acute angle that will generate the **positive numerical value**).

[**Ignore the negative sign when calculating the reference angle**]

$\cos\theta$ is equal to $-\frac{5}{7}$, and cosine is negative in quadrants 2 and 3

Quadrant 2		Quadrant 3
$\theta = 180° - \text{acute reference angle}$		$\theta = 180° + \text{acute reference angle}$
$\theta = 180° - 44{,}4153086°$	or	$\theta = 180° + 44{,}4153086°$
$\theta = 135{,}58°$		$\theta = 224{,}42°$

(c) Solve for θ if $2\tan\theta = -9$, $\theta \in (0°;360°)$

$\therefore \tan\theta = -\frac{9}{2}$

The trigonometric function must be by itself before you can find the reference angle.

Reference angle = $\tan^{-1}\left(\frac{9}{2}\right) = 77{,}47119229°$

Work with **the positive value** to find the reference angle

$\tan\theta$ is equal to $-\frac{9}{2}$ and tangent is negative in quadrants 2 and 4

Quadrant 2		Quadrant 4
$\theta = 180° - \text{acute reference angle}$		$\theta = 360° - \text{acute reference angle}$
$\theta = 180° - 77{,}47119229°$	or	$\theta = 360° - 77{,}47119229°$
$\theta = 102{,}53°$		$\theta = 282{,}53°$

EXERCISE 10

1. Solve for θ in the following. Round off your answers to 2 decimal places.

   (a) $\tan\theta = 2$, and $\theta \in (0°;360°)$
   (b) $\cos\theta = 0{,}657$, and $\theta \in (0°;360°)$
   (c) $\sin\theta = -0{,}56$, and $0° < \theta < 360°$
   (d) $3\cos\theta = -2$, and $\theta \in (0°;360°)$
   (e) $4\sin\theta - 3 = 0$, and $0° < \theta < 360°$
   (f) $2\tan\theta - 0{,}82 = 0$, and $\theta \in (0°;180°)$

2. Solve for A if $\sin A = \tan 322°$, and $A \in (0°;360°)$

FURTHER EXAMPLES

EXAMPLE 13

(a) Solve for A if $\sin(\boldsymbol{A - 24°}) = -0{,}7$ and $A \in (0°;360°)$, rounded off to two decimals.

Reference angle = $\sin^{-1}(\oplus 0{,}7) = 44{,}427004°$
(work with the positive value to find the reference angle)

$\sin(\boldsymbol{A - 24°})$ is negative in Quadrants 3 and 4

Quadrant 3		Quadrant 4
$\boldsymbol{A - 24°} = 180° + \text{acute ref. angle}$		$\boldsymbol{A - 24°} = 360° - \text{acute ref. angle}$
$\boldsymbol{A - 24°} = 180° + 44{,}427004°$	or	$\boldsymbol{A - 24°} = 360° - 44{,}427004°$
$\boldsymbol{A - 24°} = 224{,}427004°$		$\boldsymbol{A - 24°} = 315{,}572996°$
$A = 248{,}43°$		$A = 339{,}57°$

(b) Solve for A if $\cos\boldsymbol{2A} = -0{,}867$ and $2A \in (0°;360°)$, rounded off to two decimals.

Reference angle = $\cos^{-1}(0{,}867) = 29{,}88813027°$
(work with the positive value to find the reference angle)

$\cos\boldsymbol{2A}$ is negative in Quadrants 2 and 3

Quadrant 2		Quadrant 3
$\boldsymbol{2A} = 180° - \text{acute ref. angle}$		$\boldsymbol{2A} = 180° + \text{acute ref. angle}$
$\boldsymbol{2A} = 180° - 29{,}88813027°$	or	$\boldsymbol{2A} = 180° + 29{,}88813027°$
$\boldsymbol{2A} = 150{,}1118697°$		$\boldsymbol{2A} = 209{,}88813027°$
$A = 75{,}06°$		$A = 104{,}94°$

EXERCISE 11

1. Solve for θ, if $\tan(\theta+30°)=2$, and $(\theta+30°)\in(0°;360°)$
2. Solve for θ, if $\cos 3\theta = 0{,}45$, and $3\theta\in(0°;360°)$
3. Solve for θ, if $2\sin 2\theta = -0{,}56$, and $0° < 2\theta < 360°$
4. Solve for β, if $2\cos(2\beta-26°)=-1{,}5$, and $(2\beta-26°)\in(0°;360°)$

TRIGONOMETRIC EQUATIONS: GENERAL SOLUTIONS

The trigonometric functions $\sin\theta$ and $\cos\theta$ have a **period of 360°** and $\tan\theta$ has a **period of 180°**. The **period** is the number of degrees needed for a trigonometric function to complete one cycle.

This means for $\sin\theta$ and $\cos\theta$ the same numerical value will be obtained when adding or subtracting 360° to the specific angle. For $\tan\theta$, the same numerical value will be obtained when 180° is added or subtracted to the specific angle.

EXAMPLE 14

(a) Determine the general solution of $\sin\theta = \frac{1}{2}$

(b) Find θ if $\theta\in(-360°;360°)$

Solutions

(a) Reference angle = $\sin^{-1}\left(\frac{1}{2}\right) = 30°$

$\sin\theta$ is positive in Quadrants 1 and 2

Quadrant 1
θ = acute ref. angle $+\ \boldsymbol{k.360°}$ or
$\therefore \theta = 30°\ +\ \boldsymbol{k.360°}$ with $k\in Z$

Quadrant 2
$\theta = 180° -$ acute ref. angle $+\ \boldsymbol{k.360°}$
$\therefore \theta = 180 - 30°\ +\ \boldsymbol{k.360°}$
$\therefore \theta = 150°\ +\ \boldsymbol{k.360°}$ with $k\in Z$

The adding of $\boldsymbol{k.360°}$ means we are adding multiples of 360° in any direction.

(b) Select integral values for *k* and calculate specific angles in the required interval.

	$k=-1$	$k=0$	$k=1$
$\theta = 30°\ +\ \boldsymbol{k.360°}$	$-330°$	$30°$	$390°$ (Not in interval)
$\theta = 150°\ +\ \boldsymbol{k.360°}$	$-210°$	$150°$	$510°$ (Not in interval)

$\therefore \theta = -330°$ or $-210°$ or $30°$ or $150°$
$\therefore \theta\in\{-330°;-210°;30°;150°\}$

The other angles don't fall inside the interval $\theta\in(-360°;360°)$

EXAMPLE 15

(a) Determine the general solution of $\tan 2\theta = -4$

(b) Find θ if $\theta \in (-90°;180°)$

Solutions

(a) Reference angle $= \tan^{-1}(4) = 75{,}96375653°$

$\tan \mathbf{2\theta}$ is negative in Quadrants 2 and 4

Quadrant 2

$\mathbf{2\theta} = 180° - \text{acute ref. angle } \mathbf{+k.180°}$

$\mathbf{2\theta} = 180° - 75{,}96375653° \mathbf{+k.180°}$

or

Quadrant 4

$\mathbf{2\theta} = 360° - \text{acute ref. angle } \mathbf{+k.180°}$

$\mathbf{2\theta} = 360° - 75{,}96375653° \mathbf{+k.180°}$

The adding of $\mathbf{k.180°}$ means we are adding multiples of 180° in any direction.

$\mathbf{2\theta} = 104{,}0362435° \mathbf{+k.180°}$

$\theta = 52{,}02 \mathbf{+k.90°}$ with $k \in Z$

$\mathbf{2\theta} = 284{,}0362435° \mathbf{+k.180°}$

$\theta = 142{,}02° \mathbf{+k.90°}$ with $k \in Z$

(b) Find θ if $\theta \in (-90°;180°)$

	$k=-2$	$k=-1$	$k=0$	$k=1$
$\theta = 52{,}02 \mathbf{+k.90°}$	$-127{,}98°$ (Not in interval)	$-37{,}98°$	52,02°	142,02°
$\theta = 142{,}02 \mathbf{+k.90°}$	$-37{,}98$	52,02°	142,02°	232,02° (Not in interval)

$\therefore \theta = -37{,}98°$ or $52{,}02°$ or $142{,}02°$

$\therefore \theta \in \{-37{,}98°;52{,}02°;142{,}02°\}$

The other angles don't fall inside the interval $\theta \in (-90°;180°)$

Summarised notes on finding the general solution.

(a) For sine and cosine we add $\mathbf{k.360°}$ with $k \in Z$

(b) For tan we add $\mathbf{k.180°}$ with $k \in Z$

EXERCISE 12

1. Find the general solution of each of the following equations.

   (a) $\tan\theta = -2{,}5$
   (b) $\cos\theta = 0{,}4$
   (c) $\sin\theta = -0{,}12$
   (d) $\cos(\theta + 10°) = -0{,}67$
   (e) $\sin 2\theta = 0{,}522$
   (f) $3\tan 3\theta = 9$

2. (i) Find the general solution of each of the following equations.

   (ii) Find θ if $\theta \in (-180°;360°)$

   (a) $\tan 2\theta = -2{,}6$
   (b) $2\cos\theta + 0{,}66 = 0$
   (c) $\sin\left(\frac{1}{2}\theta\right) = 0{,}825$

3 Solve for θ, if $\sin(\theta+30°) = -0{,}2$, and $(\theta+30°)\in(0°;720°)$

4. Solve for θ, if $\cos(2\theta+26°) = 0{,}45$, and $\theta\in(-90°;90°)$

5. Solve for θ, if $6\tan(2\theta-10°) = -0{,}6$, and $-180° < \theta < 180°$

6. Determine the general solution for $\sin 2\alpha = -0{,}35$ where $\tan\alpha > 0$

We will now consider trigonometric equations of the form $a\sin\theta \pm b\cos\theta = 0$ and those for which factorising is necessary.

Trigonometric equations of the form: $\boldsymbol{a\sin\theta \pm b\cos\theta = 0}$

EXAMPLE 16

(a) Find the general solution to the equation $3\sin\theta = 2\cos\theta$.

(b) Solve for θ if, $\sin\theta + \cos\theta = 0$, $\theta\in(0°;360°)$

Solutions

(a) The first step is to divide both sides by $\cos\theta$ because you would like to work with a single trigonometric function.

$\therefore \dfrac{3\sin\theta}{\cos\theta} = \dfrac{2\cos\theta}{\cos\theta}$ Divide both sides by $\cos\theta$ where $\cos\theta \neq 0$

$\therefore 3\tan\theta = 2$ Use the identity $\boxed{\dfrac{\sin\theta}{\cos\theta} = \tan\theta}$

$\therefore \tan\theta = \dfrac{2}{3}$

Reference angle = $\tan^{-1}\left(\dfrac{2}{3}\right) = 33{,}69006753°$

$\tan\theta$ is positive in Quadrants 1 and 3

Quadrant 1
θ = acute ref. angle $\boldsymbol{+\ k.180°}$ or
$\theta = 33{,}69° + k.180°$ with $k\in Z$

Quadrant 3
$\theta = 180° +$ acute ref. angle $\boldsymbol{+\ k.180°}$
$\theta = 180° + 33{,}69006753° + k.180°$
$\theta = 213{,}69° + k.180°$ with $k\in Z$

(b) Solve for θ if, $\sin\theta + \cos\theta = 0$, $\theta\in(0°;360°)$

$\sin\theta + \cos\theta = 0$,

$\therefore \sin\theta = -\cos\theta$

$\therefore \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\cos\theta}{\cos\theta}$ Divide both sides by $\cos\theta$ where $\cos\theta \neq 0$

$\therefore \tan\theta = -1$ Use the identity $\boxed{\dfrac{\sin\theta}{\cos\theta} = \tan\theta}$

Reference angle = $\tan^{-1}(\oplus 1) = 45°$

$\tan\theta$ is negative in Quadrants 2 and 4

Quadrant 2 | Quadrant 4

$\theta = 180° - 45° + k.180°$ | $\theta = 360° - 45° + k.180°$

$\theta = 135° + k.180°$ with $k \in Z$ | $\theta = 315° + k.180°$ with $k \in Z$

Because the possible solutions of θ have to fall inside the interval $(0°;360°)$, we use the following table:

	$k = -1$	$k = 0$	$k = 1$
$\theta = 135° + k.180°$	–45°	135°	315°
$\theta = 315° + k.180°$	135°	315°	495°

$\therefore \theta = 135°$ or $315°$

Trigonometric equations for which factorising is necessary

EXAMPLE 17

(a) Find the general solution of $1 + \sin\theta = \cos^2\theta$

(b) Solve for θ, if $2\cos^2\theta + \cos\theta = 3$

Solutions

(a) As you can see there is a sine and cosine function in this equation. In this regard, dividing by $\cos\theta$ will not simplify the equation. The 1 creates a problem.

We will choose to work with only sine in this example because **it is easier to write** $\cos^2\theta$ **in terms of** $\sin\theta$ than writing $\sin\theta$ in terms of $\cos\theta$.

Use the identity: $\cos^2\theta = 1 - \sin^2\theta$

$\therefore 1 + \sin\theta = 1 - \sin^2\theta$ This is a quadratic trigonometric equation

$\therefore \sin^2\theta + \sin\theta = 0$ Let one side $= 0$

$\therefore \sin\theta(\sin\theta + 1) = 0$ Factorise

$\therefore \sin\theta = 0$ or $\sin\theta = -1$ Apply the zero factor law

Now we will solve each equation separately and then combine our answers at the end.

For $\mathbf{\sin\theta = 0}$: Treat the zero as a positive value and $\sin\theta$ is positive in Quadrants 1 and 2.

Reference angle = $0°$

Quadrant 1 | Quadrant 2

$\theta = 0° + k.360°$ or $\theta = 180° - 0° + k.360°$

$\theta = k.360°$ with $k \in Z$ | $\theta = 180° + k.360°$ with $k \in Z$

For $\mathbf{\sin\theta = -1}$: $\sin\theta$ is negative in Quadrants 3 and 4

Reference angle = $90°$

Quadrant 3 | Quadrant 4

$\theta = 180° + 90° + k.360°$ or $\theta = 360° - 90° + k.360°$

$\therefore \theta = 270° + k.360°$ with $k \in Z$

Therefore the final answer is:

$\therefore \theta = 180° + k.360°$ or $\theta = k.360°$ or $\theta = 270° + k.360°$ with $k \in Z$

(b) Solve for θ, if $2\cos^2\theta + \cos\theta = 3$

$2\cos^2\theta + \cos\theta - 3 = 0$ Write in standard form

$\therefore (2\cos\theta + 3)(\cos\theta - 1) = 0$ Factorise the trinomial

$\therefore 2\cos\theta = -3$ or $\cos\theta = 1$

$\therefore \cos\theta = -\frac{3}{2}$

Now we will solve each equation separately and then combine our answers at the end.

For $\cos\theta = -\frac{3}{2}$:

No solution possible because $\cos\theta$ has a minimum value of -1

For $\cos\theta = 1$: $\cos\theta$ is positive in Quadrants 1 and 4.

Reference angle = $0°$

Quadrant 1		Quadrant 4
$\theta = 0° + k.360°$	or	$\theta = 360° - 0° + k.360°$
$\theta = k.360°$ with $k \in Z$		$\theta = 360° + k.360°$ with $k \in Z$

Note that $\theta = k.360°$ and $\theta = 360° + k.360°$ will produce the same possible solutions for θ.

Final answer: $\theta = k.360°$ with $k \in Z$

EXERCISE 13

1. Find the general solutions of the following equations.
   (a) $\sqrt{3}\cos\theta - 3\sin\theta = 0$ (b) $2\cos^2\theta = \cos\theta$
   (c) $(\cos\theta + \sin\theta)(1 - 2\sin\theta) = 0$ (d) $2\sin^2\theta - \sin\theta = 1$
2. Solve for θ if $3\sin\theta - 4\cos\theta = 0$, and $\theta \in (0°;360°)$
3. Solve for θ if $2\sin^2\theta = 3\cos\theta$, and $\theta \in (-180°;180°)$
4. Solve for θ if $\frac{2\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = 1$, and $\theta \in (0°;360°)$
5. Find the general solution of $3\cos\theta + \sin\theta = 0$

Trigonometric equations: when identities are invalid

An identity is a statement of equality that is true for all values (except those values for which the identity is not defined). Previously, equations containing fractions were solved. When solving these types of equations, we always stated the restrictions to prevent division by zero.

We will now determine the restrictions for a trigonometric identity. In other words, the value(s) for which the identity will be undefined.

EXAMPLE 18

(a) Consider the identity $\dfrac{\tan x.\cos x}{\sin x}=1$.

For which values of x will the identity above be invalid? State the general solutions.

(b) Consider the identity $\tan\theta+\dfrac{\cos\theta}{1+\sin\theta}=\dfrac{1}{\cos\theta}$.

For which values of x is the identity above undefined for the interval $\theta\in(-180°;180°)$?

Solutions

(a) **First consider the tangent function:**

You will recall that the graph of $y=\tan x$ **has asymptotes** at $x=....-270°,\ -90°,\ 90°,\ 270°....$ etc. Therefore the above identity is invalid for all those values. We can summarise those answers with the following formula:

$x=90°+k.180°,\ k\in Z$

Secondly consider division by zero:

The identity will be invalid if any denominator equals zero. In this example, if $\sin x=0$. Now we solve this equation.

$\therefore$ Ref angle $=0°$

Quadrant 1 | Quadrant 2

$\therefore x=0°+k.360°,\ k\in Z$ or $x=180°+k.360°,\ k\in Z$

(b) **First consider the tangent function:**

$\therefore \theta=90°+k.180°,\ k\in Z$

Secondly consider division by zero:

$\therefore 1+\sin\theta=0$ or $\cos\theta=0$

For $1+\sin\theta=0$:

$\therefore \sin\theta=-1$

$\therefore$ Ref angle $=90°$

$\therefore$ Quadrant 3 | Quadrant 4

$\therefore \theta=270°+k.360°,\ k\in Z$ or $\theta=270°+k.360°,\ k\in Z$

For $\cos\theta=0$:

$\therefore$ Ref angle $=90°$

$\therefore$ Quadrant 1 | Quadrant 4

$\therefore \theta=90°+k.360°,\ k\in Z$ or $\theta=270°+k.360°,\ k\in Z$

*By considering what values are produced in all the answers we can see that all of these answers can be summarised using only the formula below.

$\therefore \theta=90°+k.180°,\ k\in Z$

$\therefore \theta\in\{-90°;90°\}$

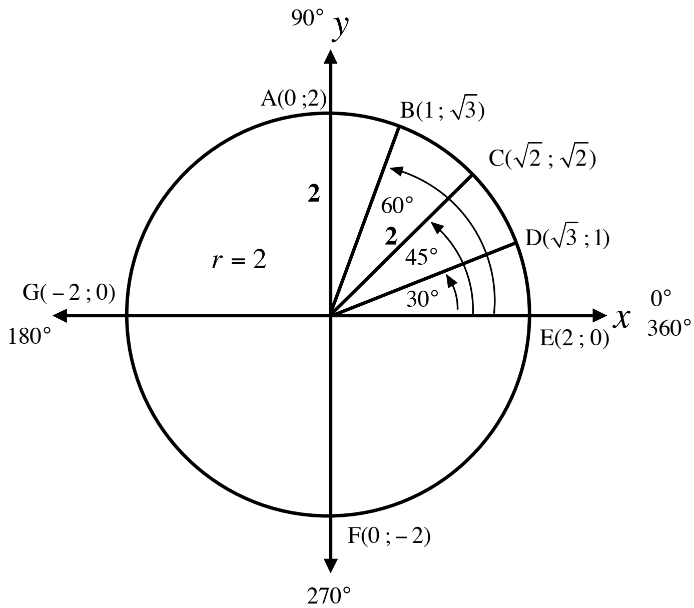
EXERCISE 14

1. For which values will the following identities be undefined? State the general solutions in each case.
   (a) $\cos\theta\left(1+\tan\theta\right)=\cos\theta+\sin\theta$
   (b) $\dfrac{1+2\sin\theta.\cos\theta}{\sin\theta+\cos\theta}=\sin\theta+\cos\theta$
   (c) $\dfrac{1}{1-\sin x}-\dfrac{1}{1+\sin x}=\dfrac{2\tan x}{\cos x}$

2. For which values of θ will the identity $\dfrac{1+\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{2}{\sin\theta}$ be undefined if $\theta\in\left(-180°;360°\right)$?

Solving Trigonometric Equations without the use of a calculator

In this section, a knowledge of special angles will be useful. The special angles are: $0°;30°;45°;60°;90°;180°;270°$ and $360°$.
The diagram for special angles is provided below.



EXAMPLE 19

Solve for θ if $\cos\theta=-\dfrac{\sqrt{2}}{2}$ and $\theta\in\left(0°;360°\right)$ without the use of a calculator.

Solution

By using the diagram above, determine the cosine of the angle that will result in a value of $\dfrac{\sqrt{2}}{2}$.

Reference angle = 45°

$\cos\theta$ is negative in Quadrants 2 and 3

Quadrant 2 or Quadrant 3

$\theta = 180° - 45° + k.360°$ or $\theta = 180° + 45° + k.360°$

$\theta = 135° + k.360°$ or $\theta = 225° + k.360°$

Because the possible solutions of θ have to fall inside the interval, $(0°;360°)$, we use the following table:

	$k=-1$	$k=0$	$k=1$
$\theta = 135° + k.360°$	$-225°$ (Not in interval)	$135°$	$495°$ (Not in interval)
$\theta = 225° + k.360°$	$-135°$ (Not in interval)	$225°$	$585°$ (Not in interval)

$\therefore \theta = 135°$ or $225°$

EXERCISE 15

1. Solve for θ without the use of a calculator if:
   (a) $\cos\theta = -\frac{1}{2}$, $\theta \in (0°;360°)$ (b) $2\sin\theta = \sqrt{3}$, $\theta \in (0°;360°)$
   (c) $\sqrt{3} + \tan\theta = 0$, $\theta \in (0°;360°)$ (d) $\cos\theta = \sqrt{3}\sin\theta$, $\theta \in (-180°;180°)$
   (e) $1 - 2\sin\theta = 0$, $\theta \in (-270°;360°)$

2. Determine the general solution of $\sin\theta = -1$ without the use of a calculator.

3. Determine the general solution of $\cos\theta = 0$ without the use of a calculator:

4. Solve for θ without the use of a calculator if
   (a) $(\cos\theta - 1)(2\sin\theta - 1) = 0$ and $\theta \in [0°;360°]$
   (b) $\sin\theta(2\cos\theta + \sqrt{3}) = 0$ and $\theta \in [0°;360°]$

5. Given: $\tan\theta = \dfrac{\sin 330°.\tan 225°}{\cos(-60°)}$ and $\theta \in (0°;360°)$
   Solve for θ without the use of a calculator.

6. (a) Sketch the graphs of $y = \sin\theta$ and $y = \cos\theta$ on separate axes for $\theta \in (-360°;360°)$.
   (b) Find the general solution of each of the following trigonometric equations using the graphs sketched in (a).
   (i) $\sin\theta = 0$ (ii) $\cos\theta = 0$
   (iii) $\cos\theta = 1$ (iv) $\sin\theta = 1$

SOLUTION OF TRIANGLES

In Grade 10 the triangles we worked with were right-angled triangles only. In Grade 11 formulae will be developed to calculate the length of a side or the size of an angle in other types of triangles. These rules include the sine-rule, area-rule and cosine-rule.
Let's begin this section by revising the work done in Grade 10.

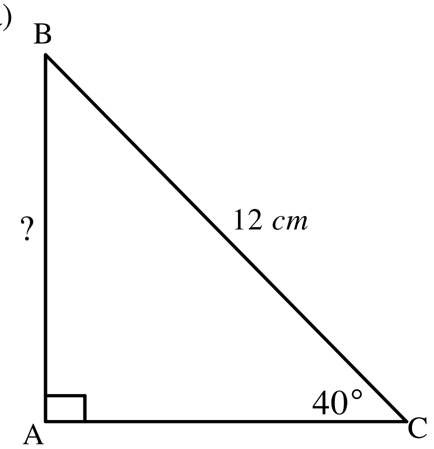
EXAMPLE 20

(a) ΔABC is given with $\hat{A} = 90°$, $\hat{C} = 40°$ and $BC = 12cm$. Determine the length of the side AB.

(b) ΔABC is given with $\hat{A} = 90°$, $AC = 7{,}71cm$ and $BC = 15cm$. Determine the size of $\hat{C}$.

Solutions

(a)



From C, AB is the opposite side, AC the adjacent side and BC the hypotenuse.

$$\sin C = \frac{AB}{BC},\ \cos C = \frac{AC}{BC},\ \tan C = \frac{AB}{AC}$$

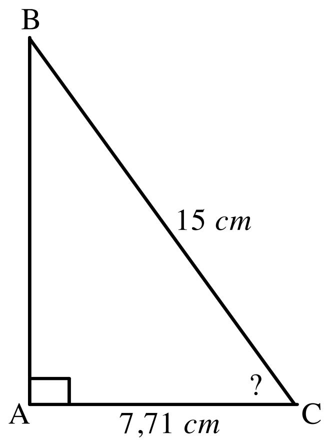
$$\therefore \sin 40° = \frac{AB}{12}\left(\frac{\text{opp}}{\text{hyp}}\right)$$

(use the trig function which has the most information)

$\therefore 12\sin 40° = AB$ LCD: 12

$\therefore AB = 7{,}71\ cm$

(b)



From C, AB is the opposite side, AC the adjacent side and BC the hypotenuse.

$$\sin C = \frac{AB}{BC},\ \cos C = \frac{AC}{BC},\ \tan C = \frac{AB}{AC}$$

$$\therefore \cos C = \frac{7{,}71}{15}\left(\frac{\text{adj}}{\text{hyp}}\right)$$

(use the trig function which has the most information)

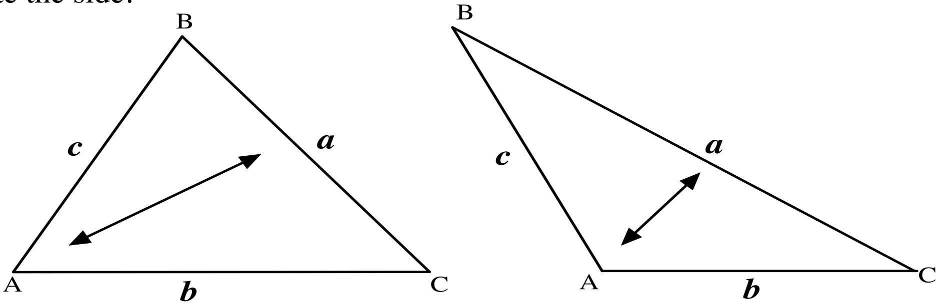
$\therefore \hat{C} = 59{,}07°$ Remember that we use the button $\cos^{-1}$ on the calculator to determine the angle.

EXERCISE 16 (It would be helpful if you sketch the diagrams)

1. If $\hat{A} = 90°$, $\hat{C} = 33°$ and $BC = 13{,}5\,cm$ determine the length of AC.
2. If $\hat{A} = 90°$, $\hat{B} = 33°$ and $BC = 7{,}35\,cm$ determine the length of AC.
3. If $\hat{A} = 90°$, $AB = 10{,}5\,cm$ and $AC = 13{,}5\,cm$ determine $\hat{C}$.
4. If $\hat{C} = 90°$, $BC = 17{,}1\,cm$ and $AB = 22{,}5\,cm$ determine $\hat{B}$.

THE SINE-RULE, COSINE-RULE AND AREA-RULE

It is important to see how triangles are labelled. For the respective **sides we use lower case letters** but more specifically, the lower case letters of the angle opposite the side.



The following pre-knowledge is of utmost importance before we prove three theorems (sine rule, cosine rule and area rule).

Consider ΔABC on the Cartesian plane with A in the standard position at the origin.

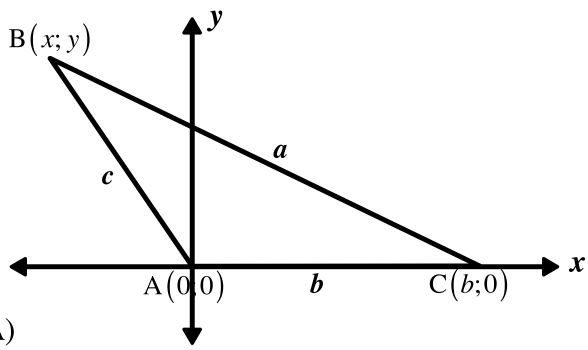
Let B have the coordinates $(x; y)$.

Then $\cos A = \frac{x}{r} = \frac{x}{c}$ and $\sin A = \frac{y}{r} = \frac{y}{c}$.

$\cos A = \frac{x}{c}$ and $\sin A = \frac{y}{c}$

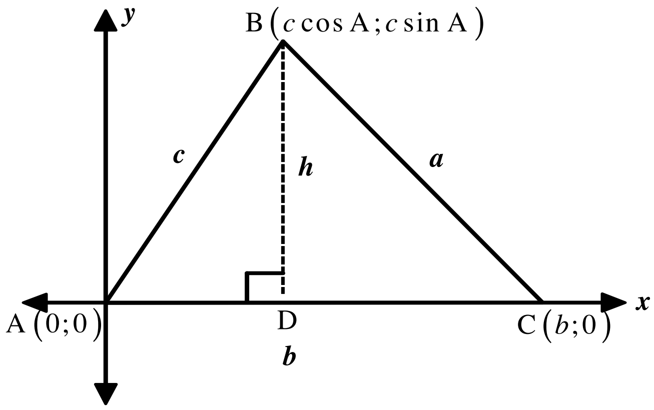
$\therefore c\cos A = x$ $\therefore c\sin A = y$

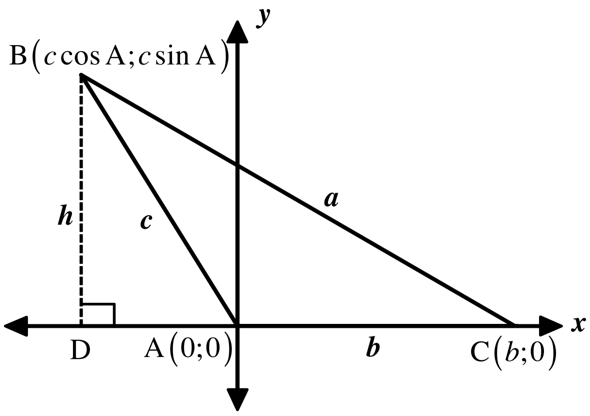
The co-ordinates of B are: $(c\cos A\,;\, c\sin A)$



THE SINE-RULE

In any ΔABC it is true that: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ or $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$





Proof

Construction: Place $\hat{A}$ in standard position and drop a perpendicular line from B to the x-axis.

If we consider AC to be the base, the y-coordinate of B is the height.

$\therefore h = c\sin A$ ….(1)

In ΔBCD we know that: $\frac{h}{a} = \sin C$

$\therefore h = a\sin C$ …..(2)

$\therefore c\sin A = a\sin C$ knowing that (1) = (2) because both are equal to h.

$\therefore \frac{\sin A}{a} = \frac{\sin C}{c}$ (÷ both sides by ac)

$\therefore \frac{a}{\sin A} = \frac{c}{\sin C}$ (Inverse of each)

Likewise, by placing either $\hat{B}$ or $\hat{C}$ in the standard position it can be proved that

$\frac{\sin B}{b} = \frac{\sin C}{c}$ or $\frac{\sin A}{a} = \frac{\sin B}{b}$ so that we may conclude that

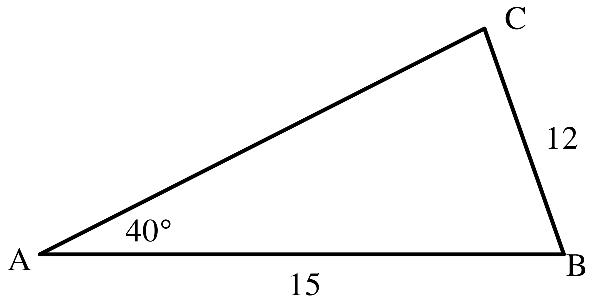
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

In the examples that follow, it will be clear that the **sine-rule is best used when two sides and one angle (not included) are known, or when two angles and one side are known.**

EXAMPLE 21

(a) In ΔABC, $\hat{A} = 40°$, $BC = 12$ and $AB = 15$. Determine the other angles of the triangle. Assume that $\hat{C}$ is acute.

KNOWN: 2 SIDES AND 1 ANGLE (not included)



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\therefore \frac{\sin 40°}{12} = \frac{\sin B}{b} = \frac{\sin C}{15}$$

$$\therefore \frac{\sin 40°}{12} = \frac{\sin C}{15}$$ Work with the pair that has the most information.

$$\therefore 15 \sin 40° = 12 \sin C$$

$$\therefore \frac{15 \sin 40°}{12} = \sin C$$

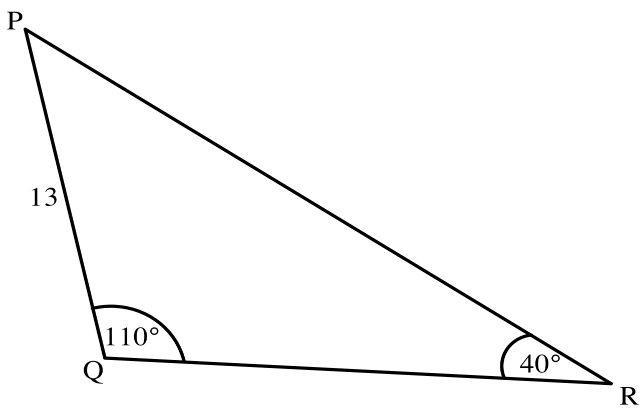
$$\therefore \sin C = 0{,}8034845$$

$$\therefore \hat{C} = 53{,}46°$$

$$\therefore \hat{B} = 180° - (40° + 53{,}46°) = 86{,}54°$$ … Angles of a Δ

(b) In ΔPQR, $\hat{Q} = 110°$, $\hat{R} = 40°$ and $PQ = 13$. Calculate the lengths of the sides.

KNOWN: 1 SIDE AND 2 ANGLES



$\hat{P} = 180° - (40° + 110°) = 30°$ … Angles of a triangle.

$$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$$

$$\therefore \frac{\sin 30°}{p} = \frac{\sin 110°}{q} = \frac{\sin 40°}{13}$$

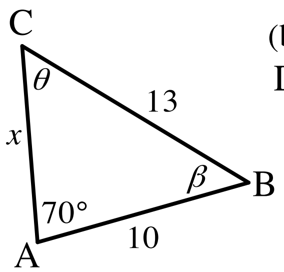
$$\therefore \frac{\sin 30°}{p} = \frac{\sin 40°}{13} \qquad \text{or} \qquad \frac{\sin 110°}{q} = \frac{\sin 40°}{13}$$

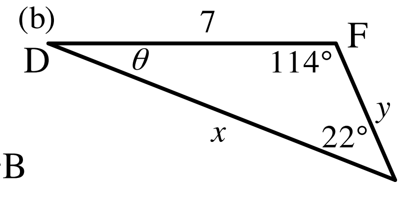
Work with the pair that has the most information.

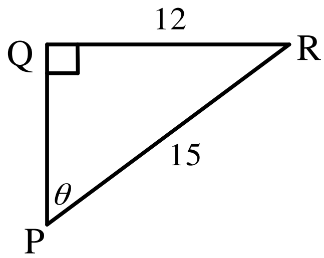
$\therefore 13\sin 30° = p\sin 40°$	$\therefore 13\sin 110° = q\sin 40°$
$\therefore \dfrac{13\sin 30°}{\sin 40°} = p$	$\therefore \dfrac{13\sin 110°}{\sin 40°} = q$
$\therefore p = 10{,}11$ units	$\therefore q = 19$ units

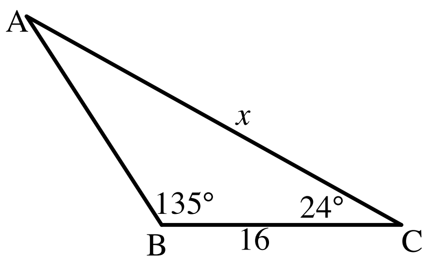
EXERCISE 17

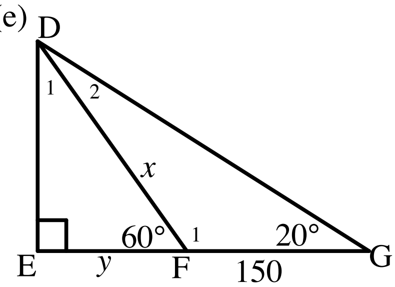
1. Determine the value of the unknowns in each case.

(a)


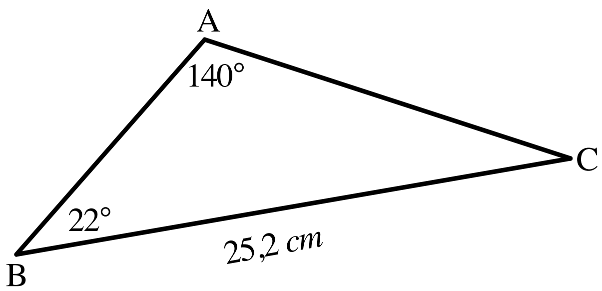
(b)


(c)


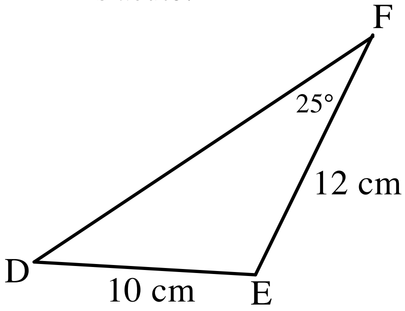
(d)


(e)


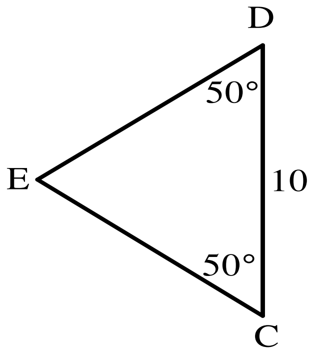
2. In the diagram below we have ΔABC with $BC = 25{,}2\,cm$, $\hat{A} = 140°$ and $\hat{B} = 22°$. Calculate the length of AB.



3. In the diagram below we have ΔDEF with $DE = 10cm$, $EF = 12cm$ and $\hat{F} = 25$. Calculate $\hat{E}$. $\hat{D}$ is acute.



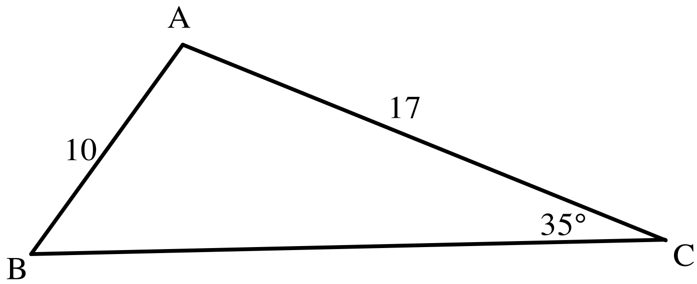
4. In the diagram below, $DC = 10$ and $\hat{D} = \hat{C} = 50°$.
   (a) Determine the length of ED.
   (b) The shortest distance from E to DC.



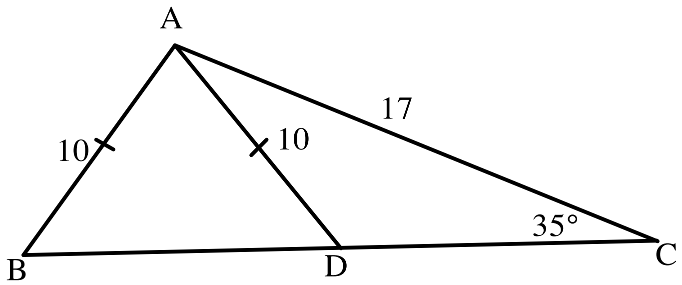
The ambiguous case

The ambiguous case occurs in triangles where two sides and one angle are given such that the given angle is opposite the shorter of the two given sides.

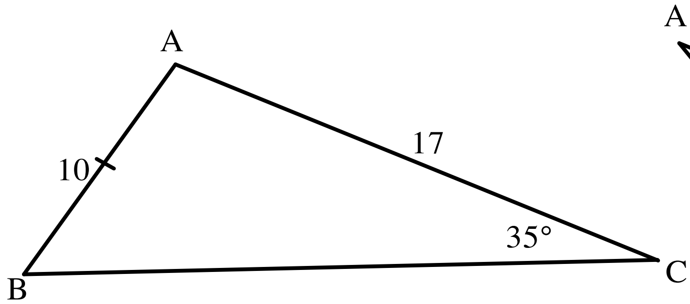
Consider the following triangle with the given dimensions.

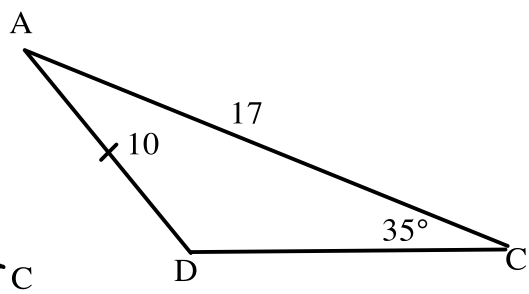


Draw a line from A to D on BC so that $AD = AB$.



Compare the newly formed ΔADC with the original ΔABC.





ΔADC and ΔABC have the same known dimensions but as you can see, $\hat{B}$ and $\hat{D}$ are not the equal. This is referred to as the **ambiguous case:** From certain given conditions, two different (non-congruent) triangles can be formed.

In ΔABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\therefore \frac{\sin A}{a} = \frac{\sin B}{17} = \frac{\sin 35°}{10}$

$\therefore 10 \sin B = 17 \sin 35°$

$\therefore \sin B = \frac{17 \sin 35°}{10}$

$\therefore \sin B = 0{,}9750799418$

$\therefore \hat{B} = 77{,}18°$ or $\hat{B} = 180° - 77{,}18°$
$= 102{,}82°$ (N/A)

(N\A because $\hat{B}$ is acute in the sketch)

In ΔADC, $\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin D}{d}$

$\therefore \frac{\sin A}{a} = \frac{\sin 35°}{10} = \frac{\sin D}{17}$

$\therefore 17 \sin 35° = 10 \sin D$

$\therefore \frac{17 \sin 35°}{10} = \sin D$

$\therefore \sin D = 0{,}9750799418$

$\therefore \hat{D} = 77{,}18°$ (N/A) or $\hat{D} = 180° - 77{,}18°$
$= 102{,}82°$

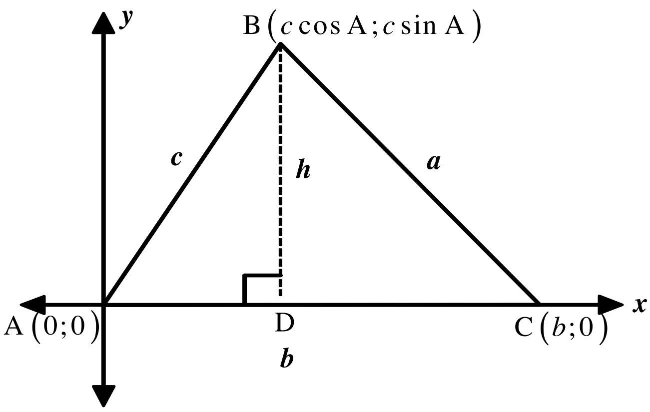
(N\A because $\hat{D}$ is obtuse)

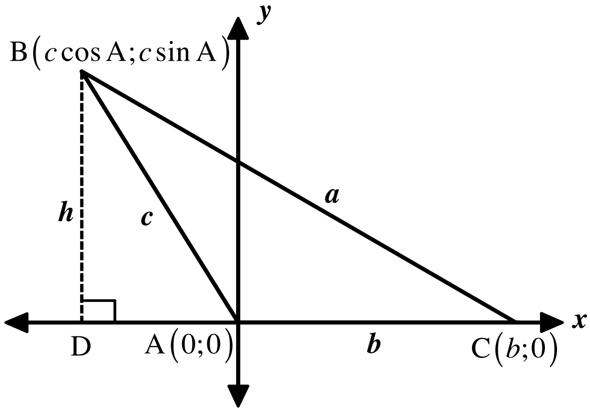
Important to note: You must always refer back to the drawn/given triangle to determine whether or not the angle in question is acute or obtuse. Especially when you are using the sine-rule.

<u>THE COSINE-RULE</u>

In any ΔABC it is true that:
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

As you can see, there are three forms of the cosine-rule. The one to use is determined by what is given or what is asked. The side you start with in the cosine-rule is opposite the angle you will use in the rule.





Proof

Construction: Place $\hat{A}$ in standard position and drop a perpendicular line from B to the x-axis.

From the distance formula we know that:

$$BC^2 = (x_B - x_C)^2 + (y_B - y_C)^2$$
$$a^2 = (c\cos A - b)^2 + (c\sin A - 0)^2$$
$$a^2 = c^2\cos^2 A - 2bc\cos A + b^2 + c^2\sin^2 A$$
$$a^2 = c^2(\cos^2 A + \sin^2 A) - 2bc\cos A + b^2 \qquad \text{Factorise}$$
$$a^2 = c^2(1) - 2bc\cos A + b^2 \qquad \cos^2 A + \sin^2 A = 1$$
$$\therefore a^2 = b^2 + c^2 - 2bc\cos A$$

Likewise,

(i) by placing $\hat{B}$ in standard position one can prove that
$b^2 = a^2 + c^2 - 2ac\cos B$

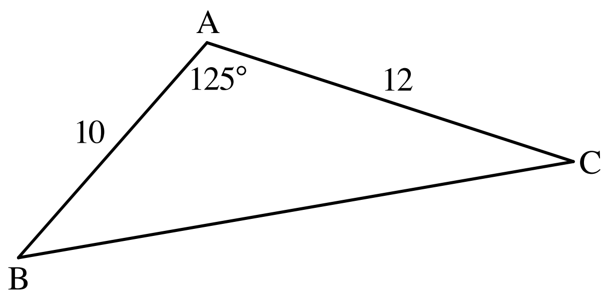
(ii) by placing $\hat{C}$ in standard position one can prove that
$c^2 = a^2 + b^2 - 2ab\cos C$

In the examples that follow it will be clear that the **cosine-rule is best used when 2 sides and 1 included angle is known, or when 3 sides are known.**

EXAMPLE 22

(a) In the diagram below we have ΔABC with $AC = 12$, $AB = 10$ and $\hat{A} = 125°$. Calculate the length of BC.

KNOWN: 2 SIDES AND 1 INCLUDED ANGLE



$a^2 = b^2 + c^2 - 2bc\cos A \qquad (BC = a)$

$\therefore a^2 = (12)^2 + (10)^2 - [2(12)(10)\cos 125°]$

The block brackets are added as a "safety precaution" to ensure that the order of calculation is done correctly.

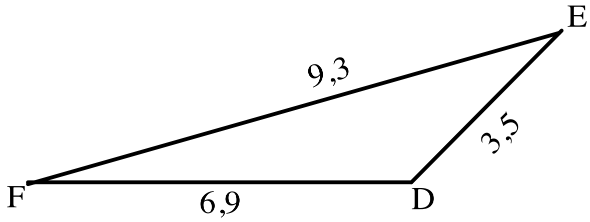
$\therefore a^2 = 381{,}6583447$

$\therefore a = 19{,}53607803$

$\therefore BC = 19{,}54$ units

(b) In the diagram below we have ΔDEF with $ED = 3,5$, $EF = 9,3$ and $FD = 6,9$. Calculate the obtuse angle $\hat{D}$.

KNOWN: 3 SIDES



$d^2 = e^2 + f^2 - 2ef\cos D$

$\therefore (9,3)^2 = (3,5)^2 + (6,9)^2 - [2(3,5)(6,9)\cos D]$

The block brackets are added as a "safety precaution" to ensure that the order of calculation is done correctly.

$\therefore 86,49 = 12,25 + 47,61 - (48,3\cos D)$

$\therefore 26,63 = -48,3\cos D$

$\therefore -0,55134... = \cos D$

$\therefore$ Reference angle $= 56,5406...°$

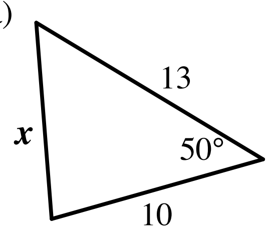
$\therefore \hat{D} = 180° - 56,5406...°$

$\therefore \hat{D} = 123,46°$ } cosine is negative in the 2nd quad.

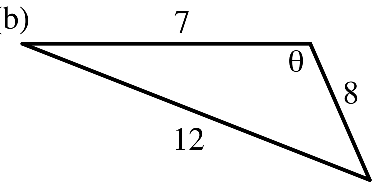
EXERCISE 18

1. Use the cosine-rule to determine the value of θ and x in the following.

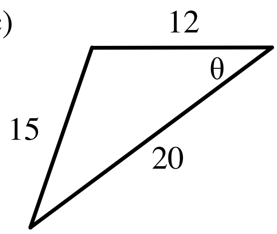
(a)



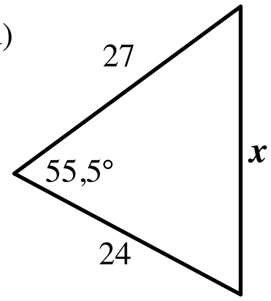
(b)



(c)

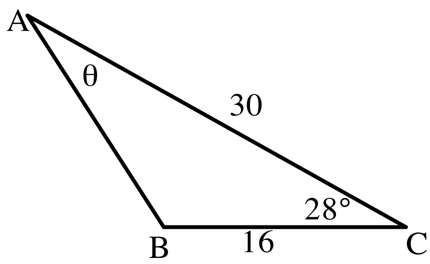


(d)

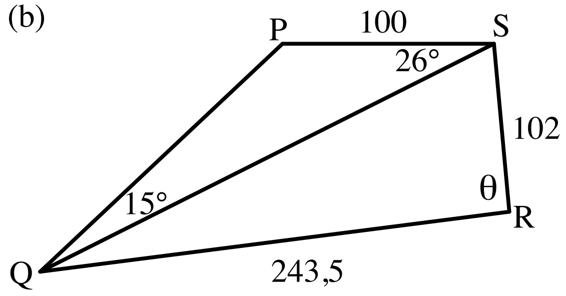


2. Calculate the value of the unknown in each case by using the sine rule or cosine rule.

(a)

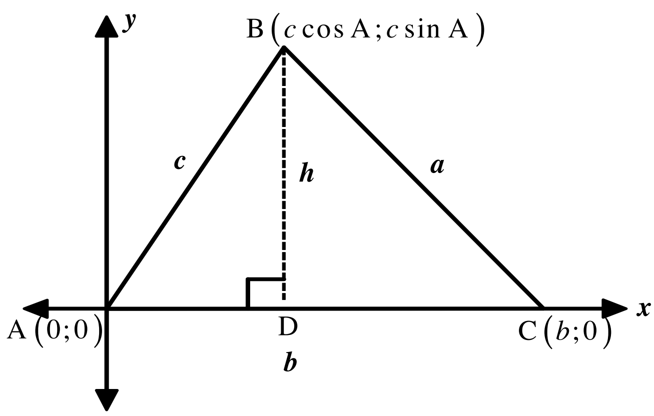


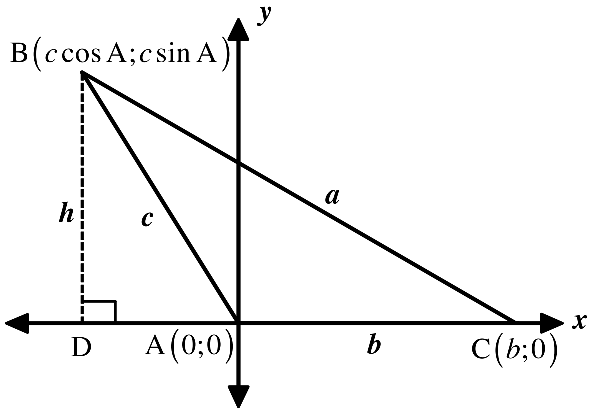
(b)



THE AREA-RULE

In any ΔABC it is true that: Area of $\Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$





Proof

Construction: Place $\hat{A}$ in standard position and drop a perpendicular line from B to the x-axis.

If we consider AC to be the base, the y-coordinate of B is the height.

$\therefore h = c\sin A$

and Area of $\Delta ABC = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times (b) \times (c\sin A)$$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}bc\sin A$$

Likewise,

(i) by placing $\hat{B}$ in standard position one can prove that:

Area of $\Delta ABC = \frac{1}{2}ac\sin B$

(ii) by placing $\hat{C}$ in standard position one can prove that:

Area of $\Delta ABC = \frac{1}{2}ab\sin C$

If you look more carefully at the formula of the area-rule and its diagram you will be able to see that the area rule can be generalised to the following:

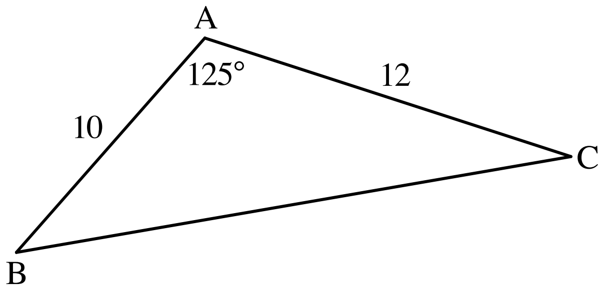
$$\text{Area of } \Delta ABC = \frac{1}{2} \times \text{side} \times \text{side} \times \sin(\text{included angle}) .$$

This means that two sides and an angle included between those sides, are used to calculate the area of a triangle.

<u>EXAMPLE 23</u>

(a) In the diagram below we have ΔABC with $AC = 12cm$, $AB = 10cm$ and $\hat{A} = 125°$. Calculate the area of ΔABC .

KNOWN: 2 SIDES AND AN INCLUDED ANGLE



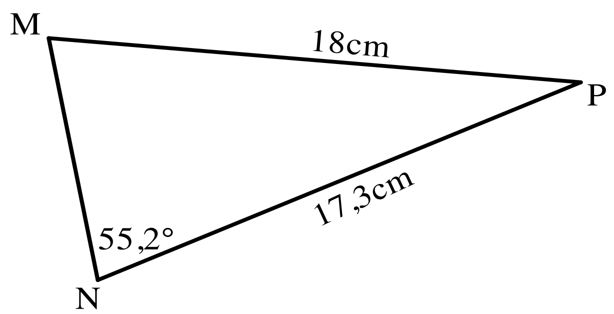
$\text{Area of } \Delta ABC = \frac{1}{2} bc \sin A$

$\therefore \text{Area of } \Delta ABC = \frac{1}{2}(12)(10)\sin 125°$

$\therefore \text{Area of } \Delta ABC = 49{,}15cm^2$

(b) In the diagram below we have ΔMNP with $MP = 18cm$, $PN = 17{,}3cm$ and $\hat{N} = 55{,}2°$. Calculate the area of ΔMNP .

KNOWN: 2 SIDES AND 1 ANGLE (not included)



From the information and the diagram it can be clearly seen that you don't have enough information to calculate the area of ΔMNP .
We have to use either the sine or the cosine-rule to find the necessary angles/sides to calculate the area of ΔMNP . Because of what is known, the sine-rule will be used.

$$\frac{\sin M}{m} = \frac{\sin N}{n} = \frac{\sin P}{p}$$

$$\therefore \frac{\sin M}{17,3} = \frac{\sin 55,2°}{18} = \frac{\sin P}{p}$$

$$\therefore \frac{\sin M}{17,3} = \frac{\sin 55,2°}{18}$$

$$\therefore 18 \sin M = 17,3 \sin 55,2°$$

$$\therefore \sin M = \frac{17,3 \sin 55,2°}{18}$$

$$\therefore \sin M = 0,7892...$$

$$\therefore \hat{M} = 52,11227109°$$

$\therefore \hat{P} = 72,68772891°$ … Angles of a triangle

$$\therefore \text{Area of } \Delta MNP = \frac{1}{2} mn \sin P$$

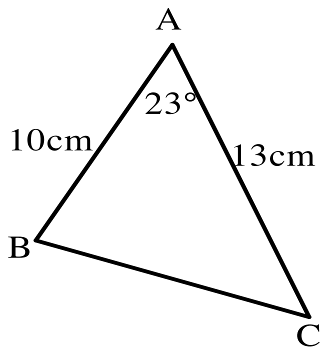
$$\therefore \text{Area of } \Delta MNP = \frac{1}{2}(17,3)(18) \sin 72,68772891°$$

$$\therefore \text{Area of } \Delta MNP = 148,65 cm^2$$

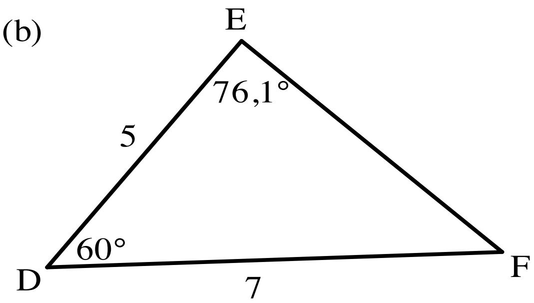
<u>EXERCISE 19</u>

1. Determine the area of the following triangles.

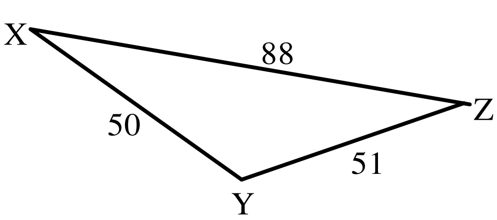
(a)



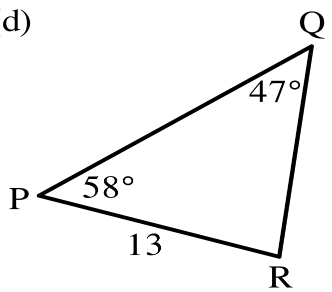
(b)



(c)

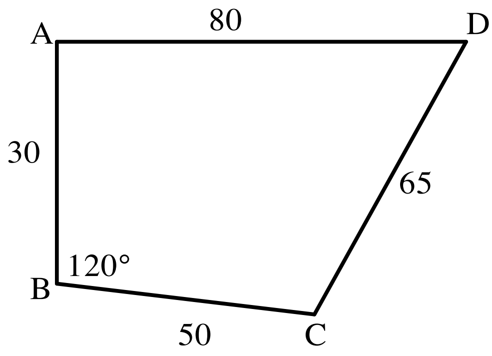


(d)

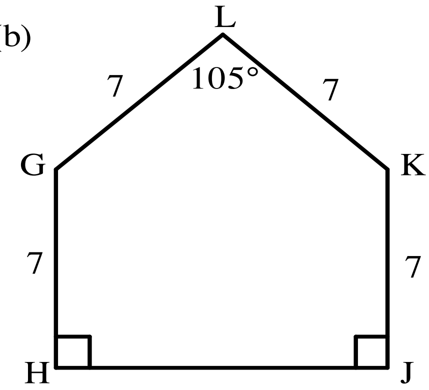


2. Determine the area of the following polygons. (**Hint: Use construction to divide the polygons into triangles where necessary.**)

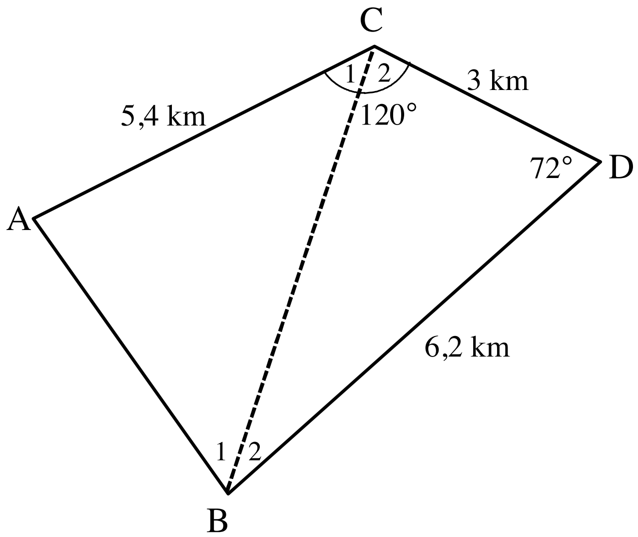
(a)



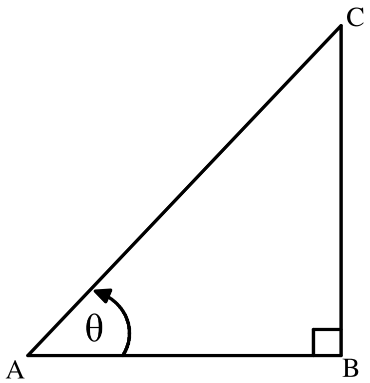
(b)

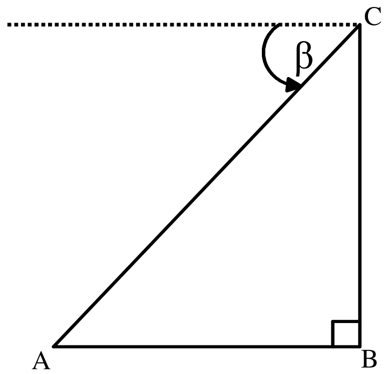


3. Consider the diagram below which represents a farmland just outside Kimberley. $\hat{C} = 120°$, $\hat{D} = 72°$ with $CD = 3\text{km}$, $BD = 6{,}2\text{km}$ and $AC = 5{,}4\text{km}$. Calculate the area of the land. (Show all steps)



Please take note of angles of elevation and angles of depression





θ is the **angle of elevation** of C from A.

β is the **angle of depression** of A from C.

Applications that consist mostly of variables only (General cases)

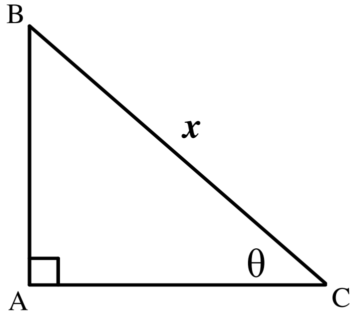
EXAMPLE 24

(a) In the diagram alongside ΔABC has a right angle at A with $BC = x$ and $\hat{C} = \theta$.

(i) Find $\hat{B}$ in terms of θ

(ii) Find the length of AB in terms of θ and x.

(iii) Find the length of AC in terms of θ and x.



Solution

(i) $\hat{B} = 180° - (90° + \theta)$

$\hat{B} = 180° - 90° - \theta$

$\hat{B} = 90° - \theta$

(ii) $\sin\theta = \dfrac{AB}{x}$

$\therefore x\sin\theta = AB$

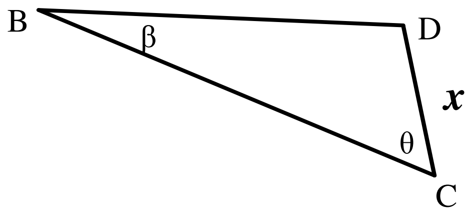
(iii) $\cos\theta = \dfrac{AC}{x}$

$\therefore x\cos\theta = AC$

(b) In the diagram alongside $D\hat{B}C = \beta$; $D\hat{C}B = \theta$ and $DC = x$ metres.

(i) Find $\hat{D}$ in terms of β and θ.

(ii) Show that $BC = \dfrac{x\sin(\beta+\theta)}{\sin\beta}$.



Solution

(i) $\hat{D} = 180° - (\beta + \theta)$ Angles of a Δ

(ii) Known: 3 angles and a side therefore we can use the **sine-rule.**

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\sin D}{d} \Rightarrow \frac{\sin\beta}{x} = \frac{\sin\theta}{c} = \frac{\sin(180° - (\beta+\theta))}{BC}$$

Work with the pair which has the most information.

$$\therefore \frac{\sin\beta}{x} = \frac{\sin(180° - (\beta+\theta))}{BC}$$

$$\therefore BC\sin\beta = x\sin(180° - (\beta+\theta))$$

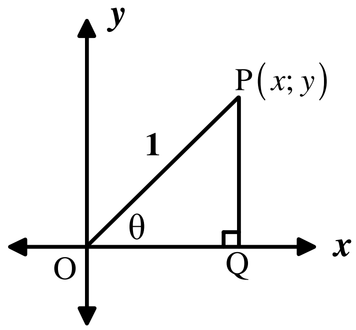
$$\therefore BC\sin\beta = x\sin(\beta+\theta)$$ Reduction formula: $(180° -)$

$$\therefore BC = \frac{x\sin(\beta+\theta)}{\sin\beta}$$

EXERCISE 20

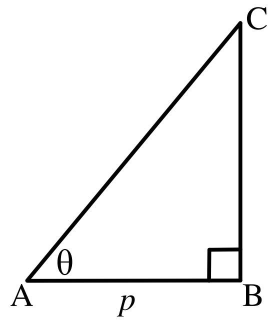
1. In the diagram alongside OP = 1 and $Q\hat{O}P = \theta$

   Write the co-ordinates of P in terms of θ



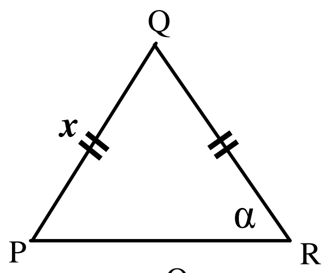
2. In ΔABC, $\hat{A} = \theta$, $\hat{B} = 90°$ and AB = p.

   Determine BC in terms of p and θ.



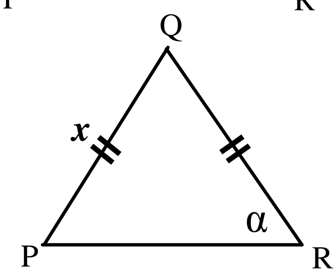
3. In the ΔPQR $\hat{R} = \alpha$ and $PQ = PR = x$.

   Use the sine-rule to show that
   $$QR = \frac{x\sin 2\alpha}{\sin\alpha}$$



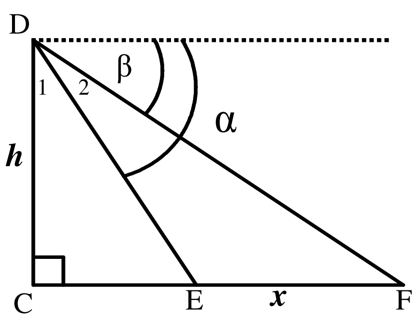
4. In the ΔPQR $\hat{R} = \alpha$ and $PQ = PR = x$

   Use the cosine-rule to show that
   $$QR^2 = 2x^2(1+\cos 2\alpha)$$



5. An observer on a cliff wishes to determine the height of the cliff h. He notices two boats on the sea. From his position (D), the angle of depression of the two boats E and F on the sea directly east of him are α and β respectively.

   The boats are x metres apart.



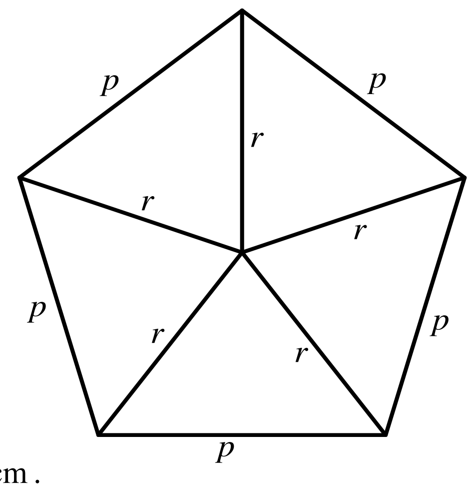
   (a) Determine the length of DE in terms h and α.

   (b) Determine $\hat{D}_2$ in terms of α and β.

(c) Show that $h = \dfrac{x\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$.

(d) Calculate the height of the cliff if $x = 250\,m$, $\alpha = 65°$ and $\beta = 30°$.

*6. Sketched alongside is a regular pentagon.



(a) Determine the area of one of the triangles in terms of r.

(b) Determine the area of the pentagon in terms of r.

(c) Show that $r = \dfrac{p}{\sqrt{2-2\cos 72°}}$

(d) Assume that the area of the pentagon equals $\dfrac{5}{2}r^2\sin 72°$.
Determine the area of the pentagon correct to the nearest cm^2 if $p = 10\,\text{cm}$.

REVISON EXERCISE ON TRIGONOMETRY

1. If $\sin\beta = \dfrac{2\sqrt{6}}{5}$ and $\beta \in (90°;270°)$ calculate, without the use of a calculator and with the aid of a diagram, the value of:

(a) $\tan\beta$ (b) $\sin(90°+\beta)$

2. Calculate the following and round your answers off to one decimal place.

(a) $\cos^2 22° - \dfrac{1}{3}\sqrt{\tan 213°}$ (b) $\sin 2\theta - \tan\dfrac{1}{2}\theta$ if $\theta = 13{,}5°$

3. Simplify the following without the use of a calculator:

(a) $\dfrac{\sin 200°}{\cos 290°}$ (b) $\sin(-150°) - \tan 840°.\cos 330°$

(c) $\tan^2 330° - \sin 120°.\tan 135°$ (d) $\sin^2 130° + \cos^2 230°$

4. Prove the following identity:

$$\tan\beta + \frac{\cos\beta}{1+\sin\beta} = \frac{\cos\beta}{1-\sin^2\beta}$$

5. Simplify without the use of a calculator:

(a) $\dfrac{\cos(540°+\theta).\tan(-\theta-180°)}{\sin(180°+\theta)}$

(b) $\dfrac{\cos(450°+\theta)}{\sin(180°-\theta)} + \cos^2(180°-\theta)$

(c) $\dfrac{\cos(90°+\theta).\cos(90°-\theta)}{(\cos 90° + \cos\theta)(\cos 90° - \cos\theta)}$

6. Find the general solution for each of the following equations:

(a) $\cos 3\theta = -0,3$ (b) $3\sin\theta = -0,3$

(c) $\tan(\theta - 10°) = \sin 225°$ (d) $\cos^2\theta = 2\cos\theta$

(e) $2\sin\theta\cos\theta = 2\sin\theta$ *(f) $2\sin\theta\cos\theta = 0$

7. Given: $4\sin 2\theta - 3\cos 2\theta = 0$

(a) Show that $\tan 2\theta = \dfrac{3}{4}$

(b) Hence, solve for θ where $\theta \in (-360°; -90°)$

8. If $\sin 28° = m$ determine the value of the following in terms of m without the use of a calculator:

(a) $\sin(-28°)$ (b) $\cos 478°$ (c) $\cos 28°$

(d) $\tan 28°$

9. (a) Prove the following identity: $1 + \dfrac{1}{\tan^2 x} = \dfrac{1}{\sin^2 x}$

(b) Determine the values for which the above identity is invalid.

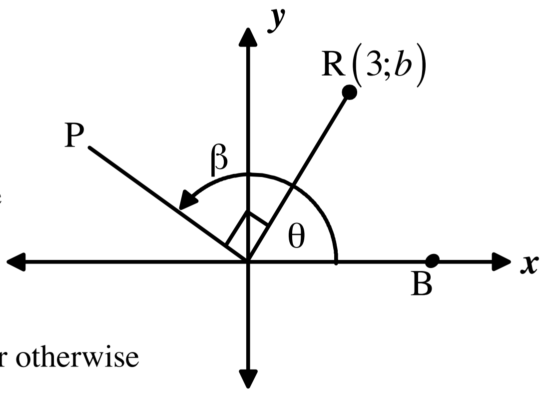
SOME CHALLENGES

1. Simplify without the use of a calculator. Show all steps.
$\cos 70°.\cos 470° - \cos(-20°).\sin(-250°)$

2. Simplify the following to a single trigonometric function of θ.
$\sqrt{1 + \tan^2\theta.\cos(-\theta).\cos(180° - \theta)}$ if $\theta \in (0°; 90°)$

3. If $\theta \in (90°; 360°)$ and $\sin\theta = \sqrt{1 - (0,624)^2}$ determine the value of $\cos\theta$ without the use of a calculator.

4. In diagram alongside $R\hat{O}B = \theta$ and $P\hat{O}B = \beta$. Point $R(3; b)$ is given.
If $\sin\beta = \dfrac{5}{13}$, calculate without the use of a calculator:



(a) $\cos\theta$

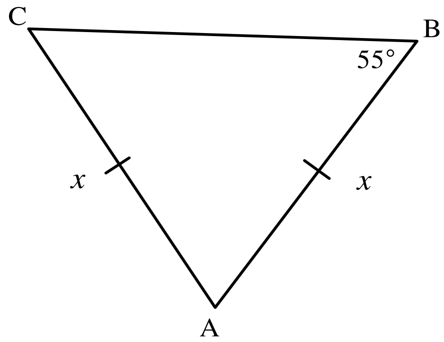
(b) the value of b and hence, or otherwise the co-ordinates of P.

5. Hiccup's calculator has two dysfunctional buttons. The $\tan\theta$ and $\cos\theta$ buttons. Help him calculate the value of $\tan 27°$ using HIS calculator.

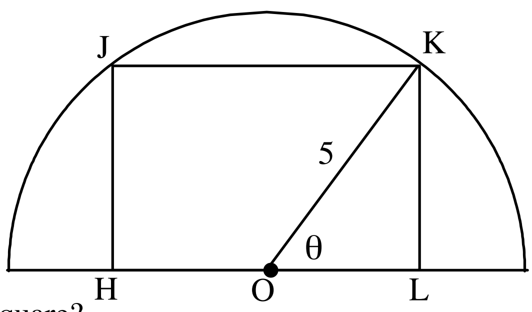
6. Show that $\tan 89° \times \tan 88° \times \tan 87° \times ..... \times \tan 1° = 1$ without the use of a calculator. Show all steps.

7. Calculate the value $\sin^2 89° + \sin^2 88° + \sin^2 87° + ... + \sin^2 2° + \sin^2 1°$ without the use of a calculator. Show all steps.

8. ΔABC has an area of $24 cm^2$ with $AB = AC = x$ and $\hat{B} = 55°$. Calculate the value of x.



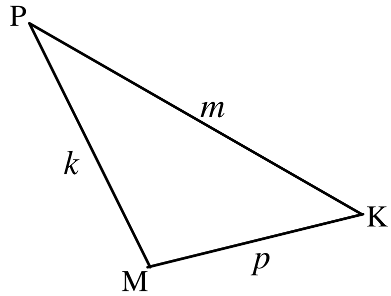
9. In the diagram below, HJKL is a rectangle in a semi-circle with centre O. The radius of the semi-circle is 5 units and $K\hat{O}L = \theta$. $HO = OL$.



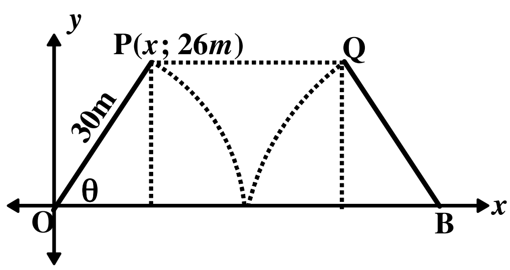
(a) If $\theta = 42°$, calculate the width of the rectangle rounded off to two decimal digits.

(b) For which value of θ will the rectangle be transformed into a square? Round off your answer to two decimal digits.

10. In the diagram $k = 2p$. If $m^2 = 10p^2$, show that ΔKPM is impossible to construct.



11. A bridge is made up of two arms, OP and BQ. The length of each arm is 30 metres. The arms drop down so that cars can cross over the bridge. Calculate the distance PQ and the size of angle θ.

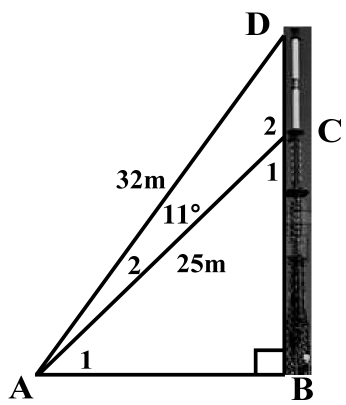


12. The diagram shows wires attached to a communications antenna. The length of each wire is 32 metres and 25 metres respectively. The angle between the wires is $11°$.

(a) Calculate the length of DC correct to the nearest whole number.

(b) Calculate the size of angle $\hat{C}_2$ correct to the nearest degree.

(c) Calculate the length of BC correct to one decimal place.



TRIGONOMETRIC FUNCTIONS

REVISION OF GRADE 10 AMPLITUDE AND VERTICAL SHIFTS

In Grade 10 you studied amplitude shifts as well as vertical shifts for the three trigonometric functions $y = a\sin x + q$, $y = a\cos x + q$ and $y = a\tan x + q$.

Amplitude is defined as follows: $\frac{1}{2} \times$ (distance between the max and min value).

For the graphs of $y = a\sin x + q$ and $y = a\cos x + q$, the amplitude is the positive value of a. The graph of $y = a\tan x + q$ has no amplitude as this graph has no maximum or minimum value.

The graphs of $y = a\sin x$, $y = a\cos x$ and $y = a\tan x$ can be shifted upwards or downwards depending on the value of q.

If $q > 0$, then the graphs shift q units upwards and if $q < 0$, the graphs shift q units downwards.

The graph of $y = a\tan x + q$ has asymptotes at $\{....; -270°; -90°; 90; 270; .....\}$

Whenever you draw this graph, make sure you always indicate the critical points. These critical points are the function values (or the y-values) at

$\{....; -315°; -225°; -135°; -45°; 45°; 135°; 225°; 315°.....\}$

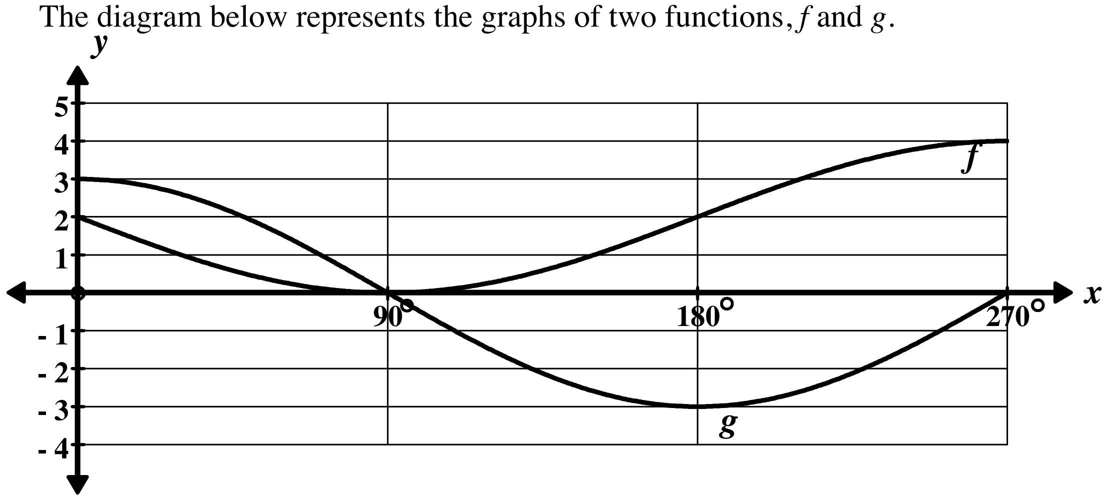
Don't forget to indicate the intercepts with axes. These points can be obtained by putting $y = 0$ in the equation of the function and solving the trigonometric equation.

EXERCISE 1

1. Sketch the graphs of each of the following functions on separate axes for the interval $x \in [0°; 360°]$. Write down the maximum and minimum value and the amplitude of each function. Indicate the coordinates of the intercepts with the axes.

(a)	$y = -3\sin x$	(b)	$y = 4\cos x$	(c)	$y = -2\cos x$
(d)	$y = 2\tan x$	(e)	$y = \frac{1}{2}\sin x$	(f)	$y = \sin x + 2$
(g)	$y = \sin x - 1$	(h)	$y = \cos x + 1$	(i)	$y = -\tan x + 1$
(j)	$y = 2\sin x - 1$	(k)	$y = 1 - 2\cos x$		

2. The diagram below represents the graphs of two functions, f and g.



(a) Write down the equation of f and g.
(b) Write down the minimum and maximum values for f and g.
(c) Write down the amplitude for f and g.
(d) Determine graphically the value of $f(180°) - g(180°)$
(e) Determine graphically the value(s) of x for which:

(1)	$g(x) \geq 0$	(2)	$g(x) < 0$	(3)	$g(x) > 0$
(4)	$f(x) = g(x)$	(5)	$f(x) > g(x)$	(6)	$f(x) = 0$
(7)	$g(x) = -3$	(8)	$f(x) - g(x) = 4$		

PERIOD SHIFTS

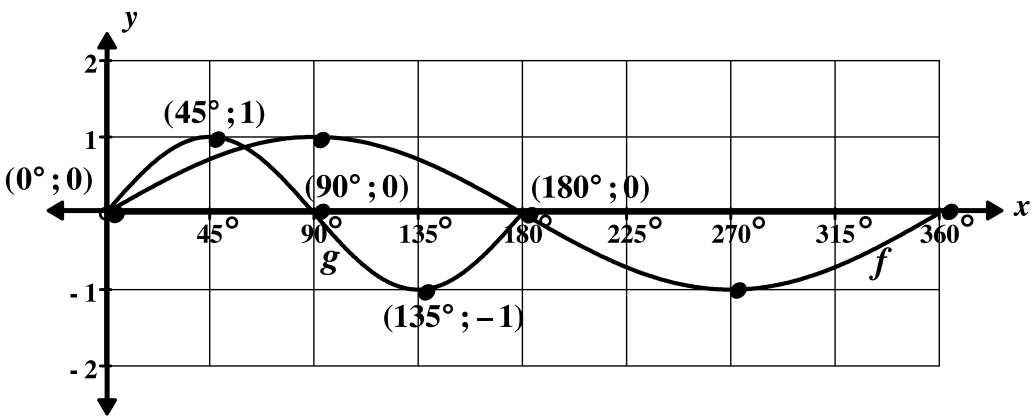
The **period** of a trigonometric function is the interval over which the graph repeats its shape. For the graph of $y = a\sin x + q$ and $y = a\cos x + q$, the period is $360°$. The graph of $y = a\tan x + q$ has a period of $180°$.
We will now consider graphs of the form $y = \sin bx$, $y = \cos bx$, $y = \tan bx$ in which the period changes.

EXAMPLE 1

Consider the graphs of $f(x) = \sin x$ and $g(x) = \sin 2x$.
The following tables have been completed and the graphs of the given functions are drawn for you on the axes provided.

x	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin x$	0	1	0	-1	0

x	$0°$	$45°$	$90°$	$135°$	$180°$
$\sin 2x$	0	1	0	-1	0



Discussion

(a) In terms of the **period**, how does the graph of $y = \sin 2x$ relate to the graph of $y = \sin x$?
The period of the graph of $y = \sin 2x$ is half of the period of $y = \sin x$
In other words, the period of $y = \sin 2x$ is $\frac{360°}{2} = 180°$
The graph of $y = \sin x$ has **critical points** at $\{0°; 90°; 180°; 270°; 360°\}$

If we now divide each critical point by 2, which is the coefficient of x in the equation $y = \sin 2x$, the critical points for the graph of $y = \sin 2x$ will be $\{0°\,;45°\,;90°\,;135°\,;180°\}$

Notice too that there are four **divisions** for the graph of $y = \sin x$:

$0°$ to $90°\,;90°$ to $180°\,;180°$ to $270°\,;270°$ to $360°$

If we divide the period of $y = \sin 2x$ by 4, we will have the new divisions for the graph of $y = \sin 2x$:

$\frac{180°}{4} = 45°$ (go up in $45°$ intervals)

$0°$ to $45°\,;45°$ to $90°\,;90°$ to $135°\,;135°$ to $180°$

(b) What is the amplitude of these graphs?
For both graphs, the amplitude is 1.

Conclusion

The **period** of the graph $y = \sin 2x$ is $\frac{360°}{2} = 180°$.

Divide the critical points of the "mother" graph $y = \sin x$ by 2 to get the new **critical points** for the graph of $y = \sin 2x$.

The **divisions** for the new graph $y = \sin 2x$ are obtained by dividing the period of $y = \sin 2x$ by 4.

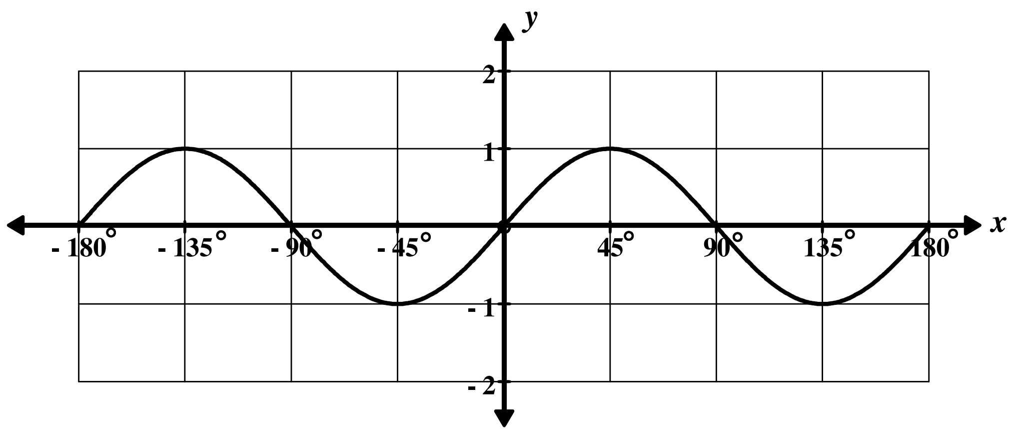
For the graph of $y = \sin 2x$, the divisions are every $\frac{180°}{4} = 45°$.

Rules for sketching graphs of the form: $y = \sin bx$, $y = \cos bx$, $y = \tan bx$

- Determine the period of the graph by dividing the period of the basic graph by b:
  For the graphs of $y = \sin bx$ and $y = \cos bx$, the period is $P = \frac{360°}{b}$ $(b > 0)$
  For the graph of $y = \tan bx$, the period is $P = \frac{180°}{b}$.
- To get the divisions for the new critical points, determine $\frac{P}{4}$.

We can sketch trigonometric graphs for intervals which include negative angles and angles greater than $360°$.

For example, the graph of $y = \sin 2x$ is drawn for the interval $x \in [-180°\,;180°]$.



EXAMPLE 2

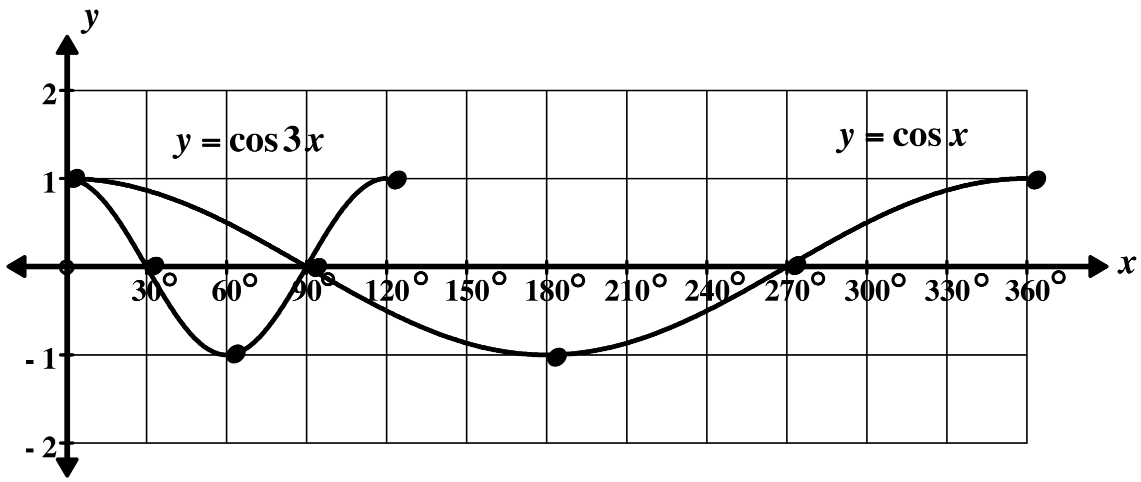
Sketch the graph of $y = \cos x$ for the interval $x \in [0°\,;360°]$. On the same set of axes, draw the graph of $y = \cos 3x$ for the interval $x \in [0°\,;120°]$.

Solution

Period of $y = \cos 3x$: $\frac{360°}{3} = 120°$

Divisions for $y = \cos 3x$: $\frac{120°}{4} = 30°$ (go up in $30°$ intervals)

Critical points for the graph of $y = \cos 3x$ are:
$\{0°\,;30°\,;60°\,;90°\,;120°\}$



Take note of the following table:

Graph	Amplitude	Maximum	Minimum	Period
$y = \cos x$	1	1	-1	360°
$y = \cos 3x$	1	1	-1	120°

EXAMPLE 3

Sketch the graph of $y = \tan 2x$ for the interval $x \in [0°\,;180°]$

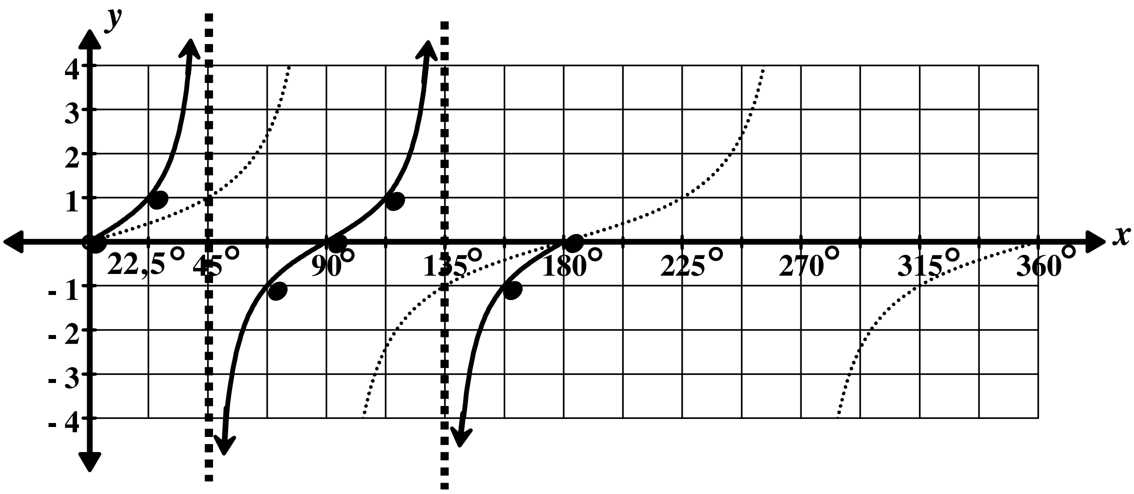
Solution

Period of $y = \tan 2x$: $\frac{180°}{2} = 90°$

Divisions for $y = \tan 2x$: $\frac{90°}{4} = 22{,}5°$ (go up in $22{,}5°$ intervals)

Critical points for the graph of $y = \tan 2x$ are:
$\{0°\,;22{,}5°\,;45°\,;67{,}5°\,;90°\,;112{,}5°\,;135°\,;157{,}5°\,;180°\}$

Asymptotes are at: $\{45°\,;135°\}$



EXERCISE 2

Sketch the graphs of each of the following functions on separate axes. Write down the maximum and minimum values, amplitude and period of each function:

(a) $y = \cos 2x$ for $x \in [-45° ; 315°]$

(b) $y = \cos \frac{1}{2}x$ for $x \in [-360° ; 720°]$

(c) $y = \sin 3x$ for $x \in [0° ; 360°]$

(d) $y = \tan \frac{1}{2}x$ for $x \in [0° ; 360°]$

(e) $y = -2\sin 2x$ for $x \in [-90° ; 180°]$

(f) $y = \frac{1}{2}\cos 2x$ for $x \in [-90° ; 180°]$

(g) $y = \sin \frac{1}{3}x + 1$ for $x \in [0° ; 1080°]$

HORIZONTAL SHIFTS

$y = \sin(x + p)$, $y = \cos(x + p)$, $y = \tan(x + p)$

These graphs work exactly like the other graphs you've studied involving shifts to the left and right.

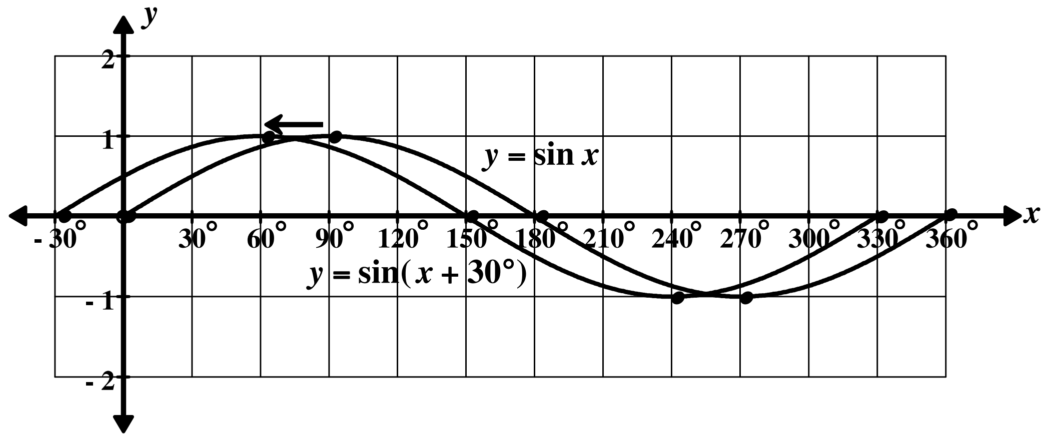
EXAMPLE 4

(a) Consider the graph of $y = \sin x$ for $x \in [0° ; 360°]$ and $y = \sin(x + 30°)$ for $x \in [-30° ; 330°]$. The following tables have been completed and the graphs of the given functions are drawn for you on the axes provided.

x	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin x$	0	1	0	-1	0

x	$-30°$	$60°$	$150°$	$240°$	$330°$
$\sin(x + 30°)$	0	1	0	-1	0

Notice that the critical points on the graph of $y = \sin x$ have been shifted $30°$ to the left. The y-values of the newly formed graph $y = \sin(x + 30°)$ are the same as the graph of $y = \sin x$.



(b) Explain how the graph of $y = \sin(x + 30°)$ relates to the graph of $y = \sin x$.

The graph of $y = \sin(x + 30°)$ is the graph of $y = \sin x$ shifted $30°$ to the left.

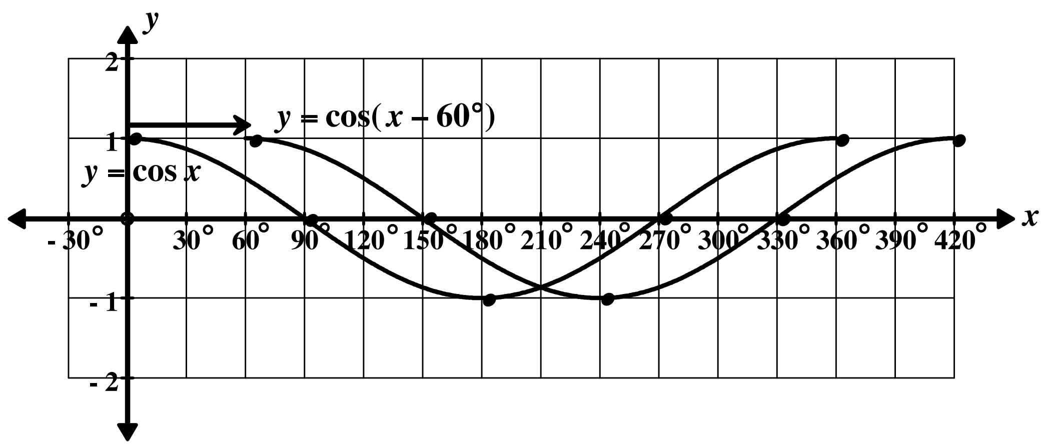
EXAMPLE 5

(a) Consider the graph of $y = \cos x$ for $x \in [0° ; 360°]$ and $y = \cos(x - 60°)$ for $x \in [60° ; 420°]$. The following tables have been completed and the graphs of the given functions are drawn for you on the axes provided.

x	$0°$	$90°$	$180°$	$270°$	$360°$
$\cos x$	1	0	-1	0	1

x	$60°$	$150°$	$240°$	$330°$	$420°$
$\cos(x - 60°)$	1	0	-1	0	1

Notice that the critical points on the graph of $y = \cos x$ have been shifted $60°$ to the right. The y-values of the newly formed graph $y = \cos(x - 60°)$ are the same as the graph of $y = \cos x$.



(b) Explain how the graph of $y = \cos(x - 60°)$ relates to the graph of $y = \cos x$.

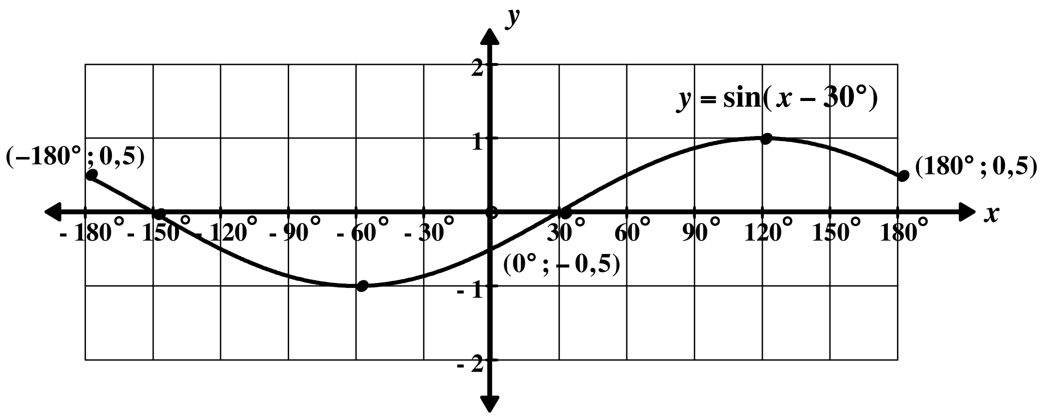
The graph of $y = \cos(x - 60°)$ is the graph of $y = \cos x$ shifted $60°$ to the right.

<u>EXAMPLE 6</u>

Sketch the graph of $y = \sin(x - 30°)$ for the interval $x \in [-180°\,;180°]$.
Indicate the co-ordinates of the end-points and intercepts with the axes.

<u>Solution</u>

The graph of $y = \sin(x - 30°)$ is obtained by shifting the graph of $y = \sin x$ $30°$ right.



The end-points can be calculated by substituting $x = -180°$ and $x = 180°$ into the equation $y = \sin(x - 30°)$. The y-intercept can be found by substituting $x = 0°$ into the equation.

For $x = -180°$:
$y = \sin(-180° - 30°) = \sin(-210°) = -\sin 210° = -(-\sin 30°) = \sin 30° = 0{,}5$

For $x = 0°$:
$y = \sin(0° - 30°) = \sin(-30°) = -\sin 30° = -0{,}5$

For $x = 180°$:
$y = \sin(180° - 30°) = \sin 150° = \sin 30° = 0{,}5$

Rules for sketching graphs of the form $y = \sin(x + p)$, $y = \cos(x + p)$, $y = \tan(x + p)$

- Sketch the "mother graph" first [$y = \sin x$, $y = \cos x$, $y = \tan x$]
- If $p > 0$ then shift the critical points on the "mother graph" p units left.
- If $p < 0$ then shift the critical points on the "mother graph" p units right.

<u>EXERCISE 3</u>

Sketch the graphs of each of the following functions on separate axes.
Write down the maximum and minimum values, amplitude and period of each function. Where necessary, calculate and indicate the coordinates of the endpoints.

(a) $y = \cos(x + 60°)$ for $x \in [-150°\,;300°]$

(b) $y = \sin(x - 45°)$ for $x \in [-135°\,;405°]$

(c) $y = \tan(x - 45°)$ for $x \in [0° ; 405°]$

(d) $y = \cos(x - 30°)$ for $x \in [-90° ; 360°]$

(e) $y = \cos(x + 30°)$ for $x \in [-180° ; 180°]$

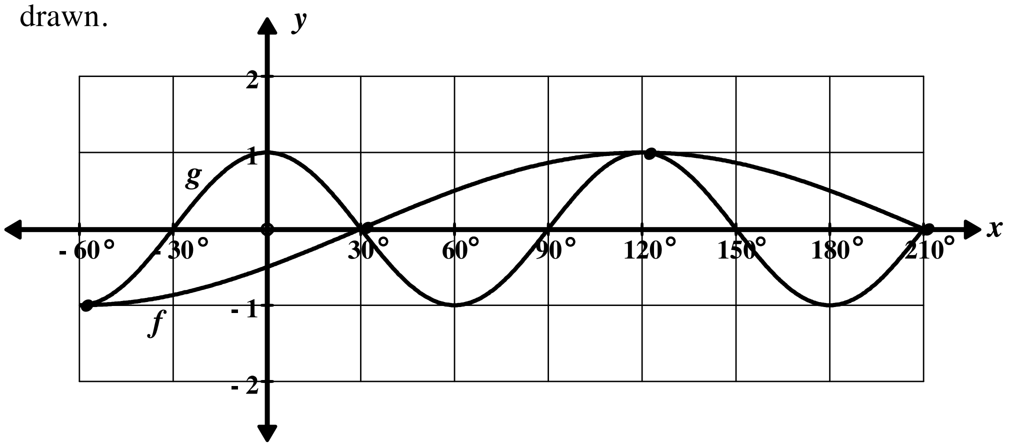
(f) $y = \sin(x - 60°)$ for $x \in [-360° ; 90°]$

<u>REVISION EXERCISE ON TRIG GRAPHS</u>

1. In each case, sketch the graphs on the same set of axes:
   (a) $y = -2\sin x$ and $y = -2 + \sin x$ for $x \in [-90° ; 180°]$
   (b) $y = \cos 3x$ and $y = 3\cos x$ for $x \in [0°; 360°]$
   (c) $y = -2\sin(x - 30°)$ and $y = \frac{1}{2}\cos(x + 30°)$ for $x \in [-30° ; 390°]$
   (d) $y = \tan x - 1$ and $y = 2\tan x$ for $x \in (-90° ; 180°)$

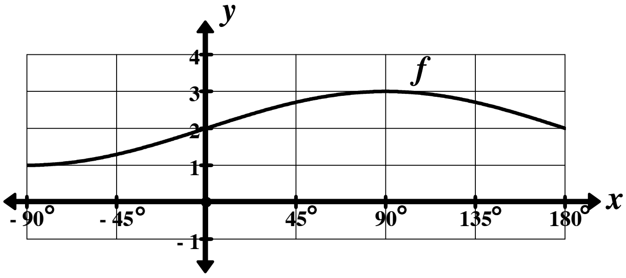
2. (a) Sketch the graph of $f(x) = \cos 2x$ for $x \in [0°; 180°]$
   (b) If the graph of f is shifted $45°$ to the right, which of the following equations will represent the equation of the newly formed graph?
   (1) $y = \cos(2x - 45°)$ (2) $y = \cos(2x + 45°)$
   (3) $y = \cos(2x - 90°)$ (4) $y = \cos(2x + 90°)$
   (c) Sketch the newly formed graph on a set of axes for $x \in [0°; 225°]$
   (d) Write down an alternative equation for the newly formed graph.

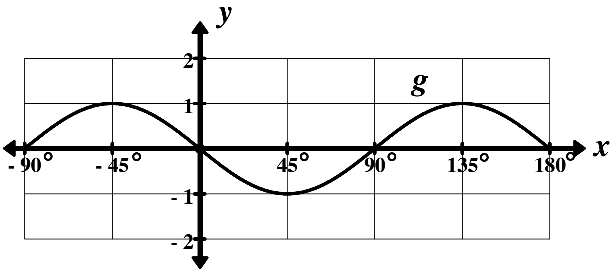
3. In the diagram below, the graphs of two trigonometric functions are drawn.

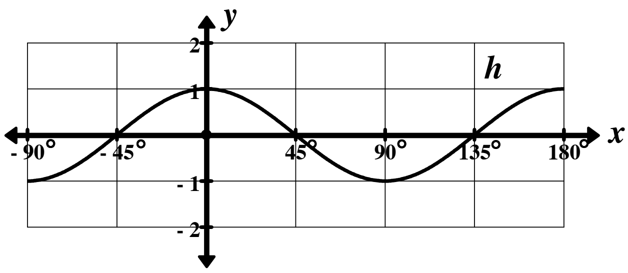


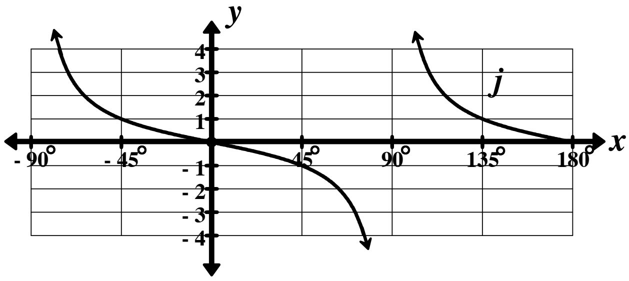
   (a) Write down the equation for the graph of f.
   (b) Write down the equation for the graph of g.
   (c) Determine graphically the value(s) of x for which:
   (1) $f(x) = g(x)$ (2) $f(x) < g(x)$
   (3) $f(x) > g(x)$ (4) $f(x) \geq g(x)$
   (5) $f(x).g(x) \leq 0$ (6) $f(x).g(x) \geq 0$
   (7) $f(x) = 0$ (8) $f(x) \geq 0$
   (9) $g(x) = 0$ (10) $g(x) < 0$
   (d) Write down the period of the graph of $y = g\left(\frac{1}{6}x\right)$

4. The graphs of four different trigonometric functions are represented below.



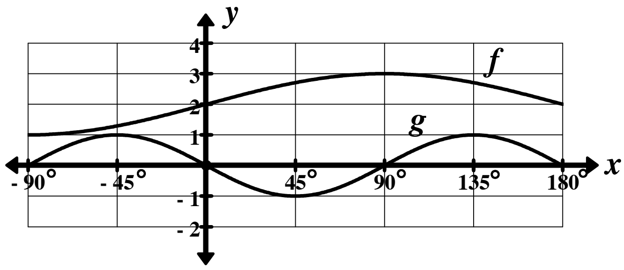


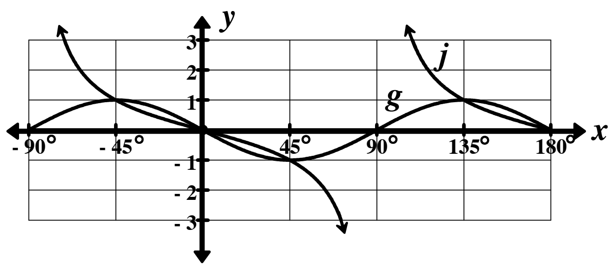




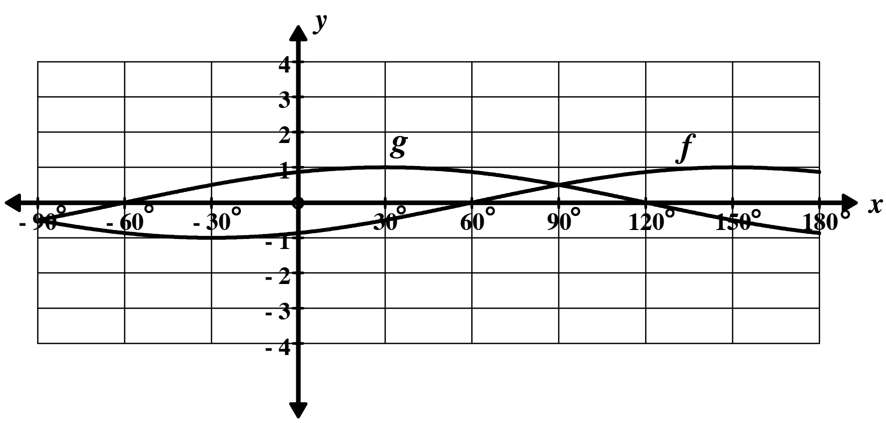
(a) Determine the equation of each graph.
(b) Write down the amplitude of graph *f*.
(c) Write down the period of graph *f* and *g*.
(d) Determine graphically the values of *x* for which:
   (1) the graph of *h* increases.
   (2) the graph of *j* decreases.

5. Determine graphically the values of *x* for which:
(a) $f(x) - g(x) = 2$
(b) $g(x) \geq j(x)$





6. Determine the equation of *f* and *g*



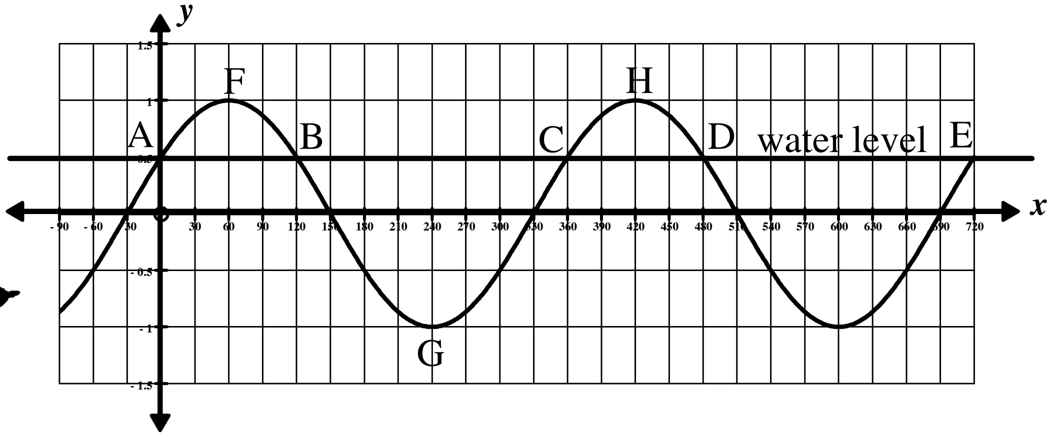
SOME CHALLENGES (MATHEMATICAL MODELLING PROBLEMS)

Simplifying a real-life problem into a model which can be solved mathematically is called **mathematical modelling**. An example of mathematical modelling is estimating the height of a large waterfall using trigonometry.

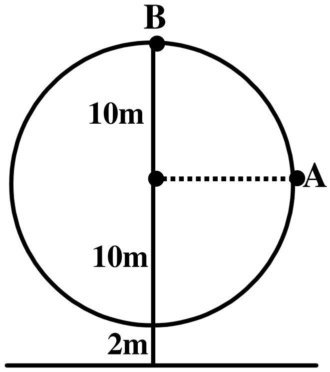
1. A river estuary is formed where a river meets the sea. The image shows the Knysna Bay River Estuary in the Southern Cape. The height of the tide, h metres, in this estuary is modelled by the function $h = 3\sin 20t + 10$ where t is the number of hours after midnight. [source: www.nmmu.ac.za/cerm/cerm3.html].

   (a) What was the height of the tide at midnight on Monday?
   (b) When is the height of the tide 8 metres for the first time, starting from midnight on Monday?
   (c) When is the height of the tide 8 metres for the second time, starting from midnight on Monday?

2. Marine biologists in Durban want to monitor the pathway of a dolphin in and out of the water using equipment which measures the horizontal movement of the dolphin in degrees (referred to as polar coordinates). The vertical movement relative to the water level is measured in metres on the vertical y-axis. The equation of the dolphin's movement is defined by the equation $y = \sin(x + 30°)$. The movement is monitored for the interval between A and E.
   (a) Determine the height of the water level above the x-axis.
   (b) Calculate the co-ordinates of the points where the dolphin leaves or enters the water (A, B, C, D and E).
   (c) Determine the distance below the water level at which the dolphin is the deepest (G).

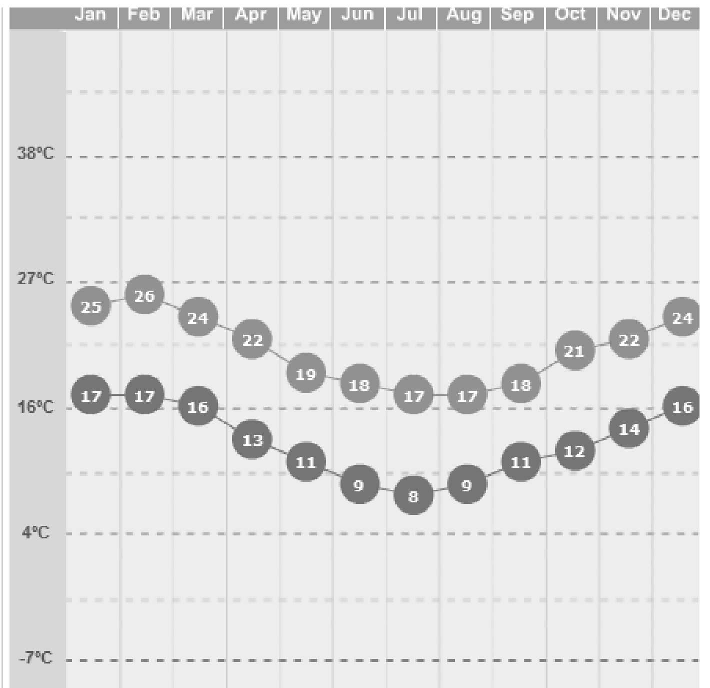


3. At Gold Reef City in Johannesburg, there is a ferris wheel called the Giant Wheel as shown in the image alongside. The diameter of the wheel is 20 metres. Its centre is 12 metres above the ground. Before allowing the public to use the wheel, the wheel is tested using one chair. The wheel rotates anti-clockwise and is timed for each revolution once it reaches its working speed. The timing starts as the chair moves upwards through the point in line with the centre of the wheel (A). The wheel rotates once every 36 seconds. B is the highest point reached by the chair. The equation $y = a + b\sin(10t)^\circ$ is a suitable model for the height, y metres, of the chair above the ground t seconds after timing starts.



(a) Calculate the value of a (you may assume that $t = 0$ at A).
(b) Calculate the value of b.
(c) Calculate the times during the first minute when the chair is 17 metres above the ground.
(d) Sketch the graph of y against t for $0 \leq t \leq 54$

4. Cape Town is one of the most beautiful cities in the world and one of the cities in South Africa that hosted the Soccer World Cup. The weather in Cape Town varies considerably over the year. The following graph represents the average monthly high and low temperatures over a year in Cape Town (source: www.weather.com).
(a) Find the amplitude of each graph.
(b) Find the vertical shift of each graph (Draw in the x-axis at $0°C$).
(c) Would the sine or cosine function represent each graph best? Explain.
(d) Determine an equation i.e. find a suitable mathematical model for each function.

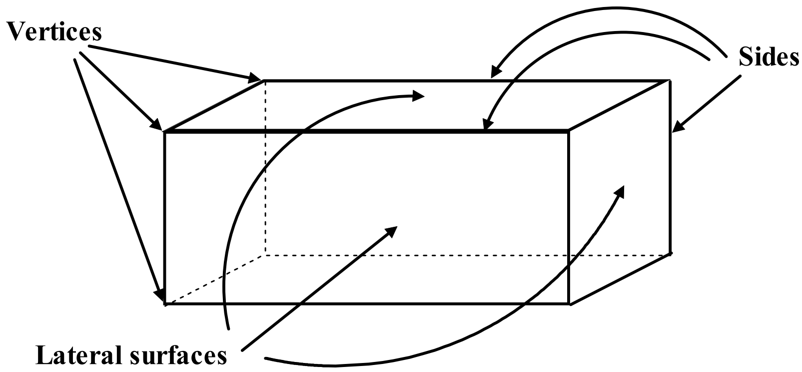


CHAPTER 7 – MEASUREMENT

<u>REVISION OF GRADE 10 CONCEPTS</u>

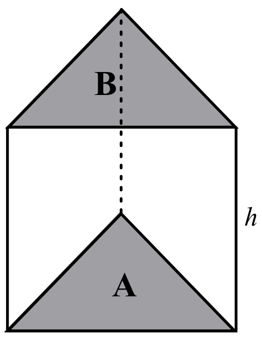
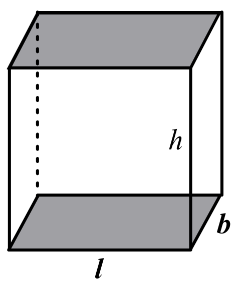
<u>POLYHEDRONS</u>

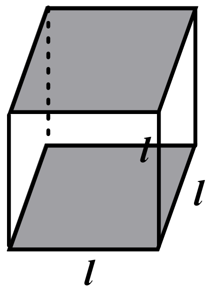
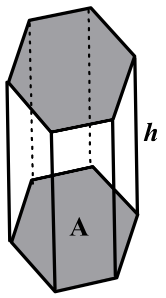
A polyhedron is a three-dimensional figure with four or more flat surfaces.

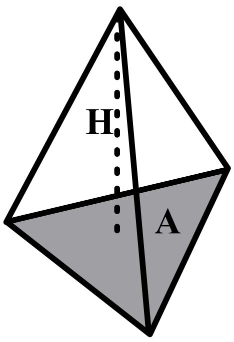
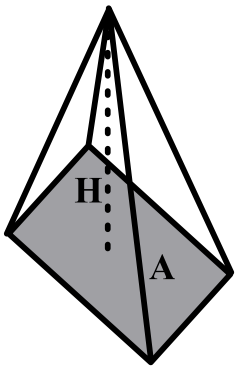


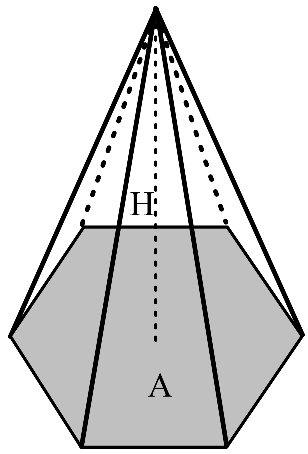
There are three types of polyhedrons:

- **Prisms:**
  A prism is a polyhedron in which the top and bottom surfaces are parallel to one another and any section parallel to the base is congruent with the base.
- **Pyramids:**
  A pyramid is a polyhedron with a polygon as base. Three or more triangles are based on the sides of the polygon and meet in one point, the apex of the pyramid. Right pyramids are such that the apex is perpendicularly above the centre of the base. The slanted triangles will therefore be congruent.
- **Regular polyhedrons (Platonic Solids):**
  A regular polyhedron is a polyhedron in which all the lateral surfaces are the same regular polygon and where equal numbers of sides meet at each vertex. The sides are all equal in length and the angles are all equal.

• Prisms	Name	Volume	Surface Area
	**Triangular Prism**	Area of base $(\Delta A)\times h$	Sum of the areas of triangles A and B and three rectangles.
	**Cuboid or Rectangular Prism**	Area of base $\times h$ $= l\times b\times h$	Sum of the areas of six rectangles $2lb+2lh+2bh$

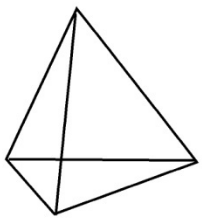
	**Cube**	Area of base $\times h$ $= l \times l \times l$ $= l^3$	Sum of the areas of six squares $= 2l^2 + 2l^2 + 2l^2$ $= 6l^2$
	**Hexagonal Prism**	Area of base $\times h$	Sum of the areas of six rectangles and two hexagons.

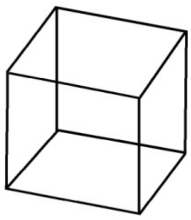
• **Right Pyramids**	**Name**	**Volume**	**Surface Area**
	**Triangular pyramid**	$\frac{1}{3}(\text{A} \times \text{H})$ where A = area of base H = height	Sum of the area of the base and three triangles.
	**Rectangular pyramid**	$\frac{1}{3}(\text{A} \times \text{H})$ where A = area of base H = height	Sum of the area of the base and four triangles.

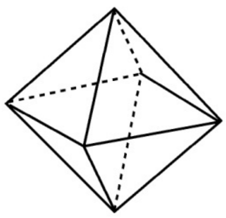
	**Hexagonal pyramid**	$\frac{1}{3}(A \times H)$ where A = area of base H = height	Sum of the area of a hexagon and six triangles.

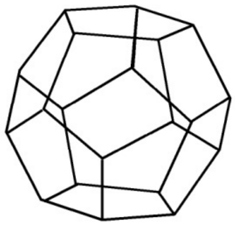
- **<u>Regular polyhedrons (The five platonic solids)</u>**

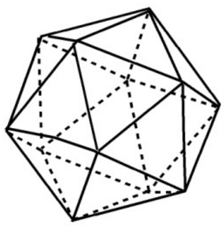
Tetrahedron: is a regular polyhedron with 4 regular surfaces.
Hexahedron (cube): is a regular polyhedron with 6 regular surfaces.
Octahedron: is a regular polyhedron with 8 regular surfaces.
Dodecahedron: is a regular polyhedron with 12 regular surfaces.
Icosahedron: is a regular polyhedron with 20 regular surfaces.

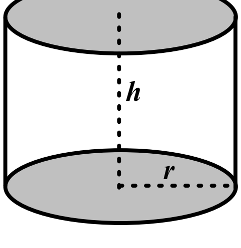

tetrahedron

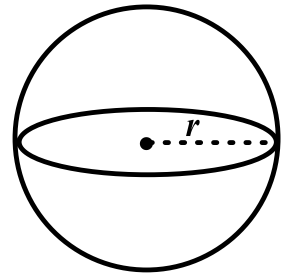
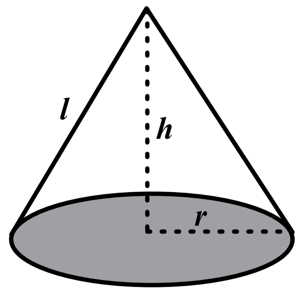

cube


octahedron


dodecahedron

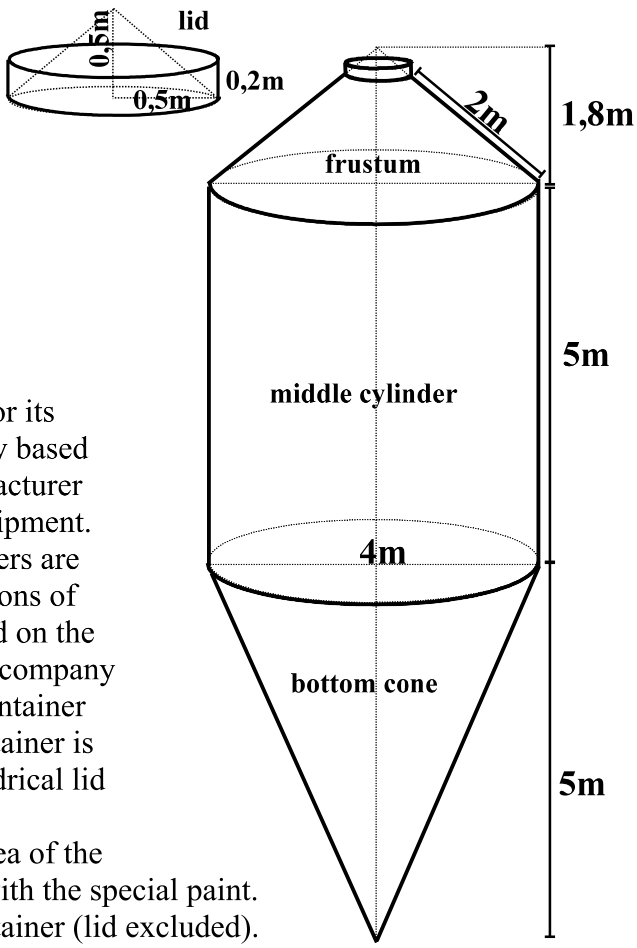

icasahedron

Other 3d solids	Name	Volume	Surface Area
	**Cylinder (only prism that is not a polyhedron)**	$\pi r^2 h$	$2\pi r^2 + 2\pi rh$

	**Sphere (not a prism nor a polyhedron)**	$\frac{4}{3}\pi r^3$	$4\pi r^2$
	**Cone (not a prism nor a polyhedron)**	$\frac{1}{3}\pi r^2 h$	$\pi rl + \pi r^2$

In this chapter, we will revise the surface area and volume of:
Rectangular prisms, triangular prisms, cylinders, triangular pyramids, rectangular pyramids, cones and spheres and combinations of these figures.

<u>**EXAMPLE**</u>

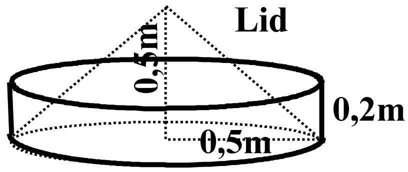


South Africa is known internationally for its superb exports. For example, a company based in Johannesburg is a world-class manufacturer and provider of agricultural storage equipment. Two large upright grain storage containers are shown in the image above. The dimensions of the upright large containers are provided on the diagram alongside. To prevent rust, the company must paint the exterior surface of the container with a special rust-proof paint. The container is hollow inside and has a screw-off cylindrical lid on top.

(a) Calculate the exterior surface area of the container that must be painted with the special paint.

(b) Calculate the volume of the container (lid excluded).

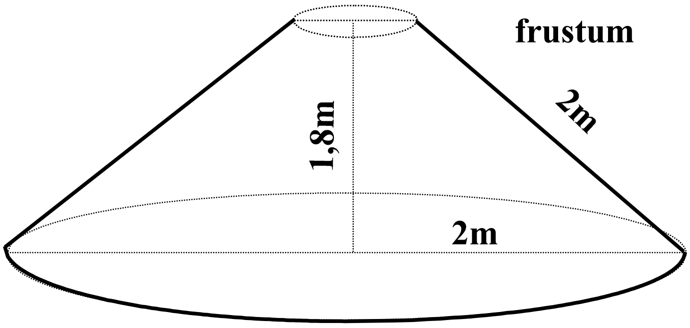
(a) The **cylindrical lid** is open at the bottom.
The formula for the surface area is therefore

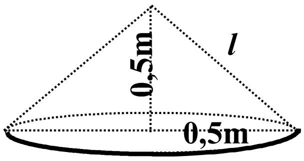


Surface area $= \pi r^2 + 2\pi rh$ (bottom circle missing)

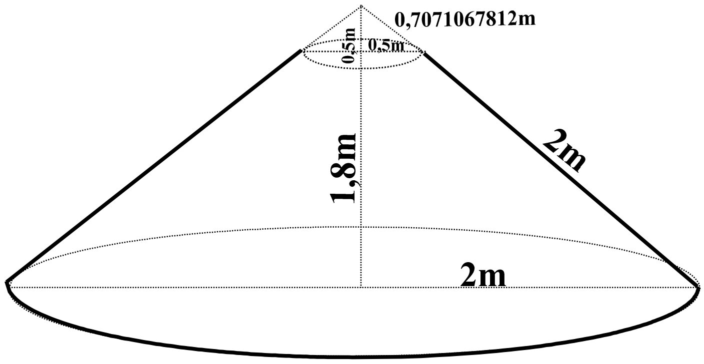
$\therefore$ Surface area $= \pi(0,5)^2 + 2\pi(0,5m)(0,2m) = \frac{9\pi}{20} m^2$

A **frustum** is a cone with the end cut off. In order to calculate the surface area of this frustum, you will need to first calculate the area of the large cone and then the area of the smaller cone which has been cut off at the top. The difference between the two surface areas will be the surface area of the frustum.





small cone



large cone (with small top cone)

Surface area of smaller cone
(top cut-off piece) $= \pi rl$
(cone is open at the bottom so the area of the bottom circle is removed from the formula $\pi rl + \pi r^2$)
We need to calculate the slant height (l):

$l^2 = (0,5)^2 + (0,5)^2$

$\therefore l^2 = 0,5$

$\therefore l = 0,7071067812m$

Surface area of smaller cone
$= \pi(0,5m)(0,7071067812m)$

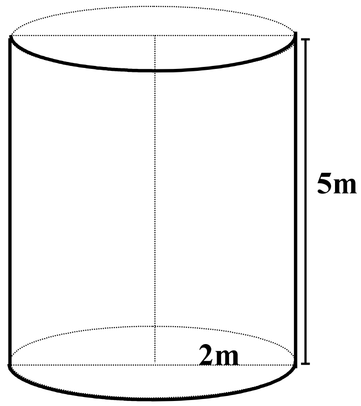
$= 1,110720735m^2$

Surface area of large cone $= \pi rl$
(cone is open at the bottom so the area of the bottom circle is removed from the formula $\pi rl + \pi r^2$)
The slant height is $2,7071067812m$
Surface area of large cone
$= \pi(2m)(2,7071067812m)$

$= 17,009253355m^2$

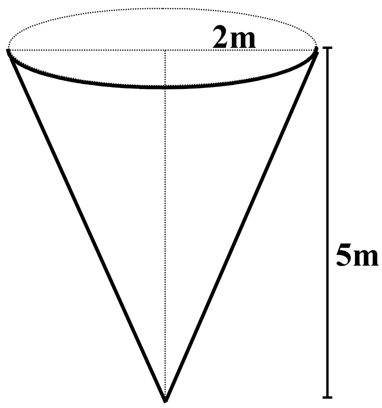
Surface area of frustum
$=$ surface area of large cone $-$ surface area of smaller cone
$= 17,009253355m^2 - 1,110720735m^2$

$= 15,89853282m^2$



The **middle cylinder** is open at the top and bottom (two circles are removed from the formula $2\pi rh + 2\pi r^2$)

Surface area $= 2\pi rh = 2\pi(2m)(5m) = (20\pi)m^2$



The surface area of the **bottom cone** is

$= \pi(2m)(5m) = (10\pi)m^2$

(open on top)

Total exterior surface area of container

$= \frac{9\pi}{20}m^2 + 15,89853282m^2 + (20\pi)m^2 + (10\pi)m^2$

$= 111,6m^2$

(b) Volume of frustum = volume of large cone − volume of small cone

$= \frac{1}{3}\pi(2m)^2(1,8m + 0,5m) - \frac{1}{3}\pi(0,5m)^2(0,5m)$

$= \frac{1}{3}\pi(2m)^2(2,3m) - \frac{1}{3}\pi(0,5m)^2(0,5m)$

$= \frac{121\pi}{40}m^3$

volume of middle cylinder $= \pi(2m)^2(5m) = (20\pi)m^3$

volume of bottom cone $= \frac{1}{3}\pi(2m)^2(5m) = \frac{20\pi}{3}m^3$

Total volume of container

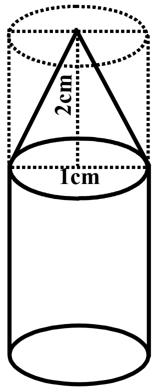
$=$ vol of frustum + vol of middle cylinder + vol of bottom cone

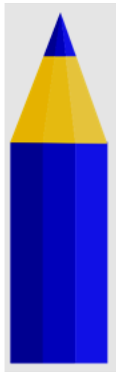
$= \frac{121\pi}{40}m^3 + (20\pi)m^3 + \frac{20\pi}{3}m^3$

$= 93,3m^3$

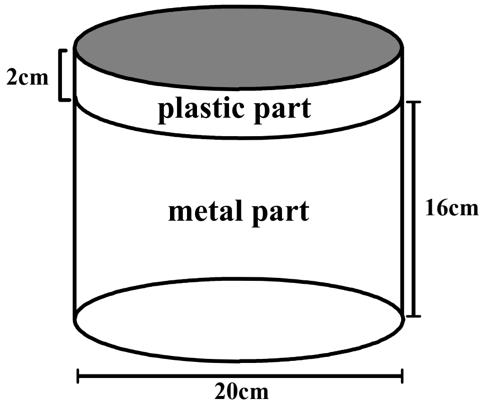
REVISION EXERCISE

1. One of the ends of a cylindrical pencil is to be sharpened to produce a perfect cone. The length of the pencil is not to be lost. The diameter of the pencil is 1 cm and the length of the cone is 2 cm. Calculate the volume of the shavings removed.

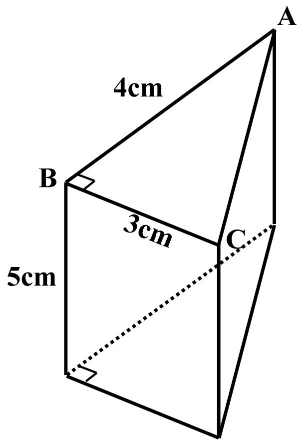




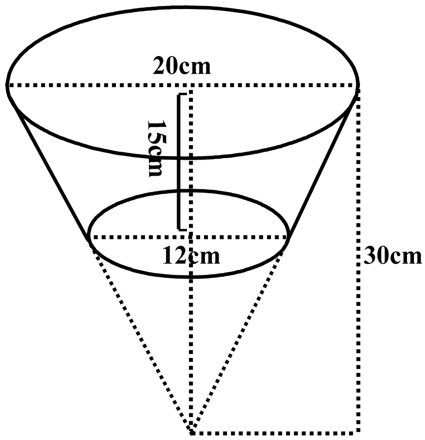
2. A cylindrical coffee container made of a metal part and a plastic lid is shown below. The dimensions of the container are indicated.
   (a) Calculate the volume of the metal part of the tin.
   (b) Calculate the surface area of the metal part of the coffee tin.
   (c) If the height of the plastic lid is doubled, calculate the volume of the plastic lid. How does this affect the original volume of the lid?



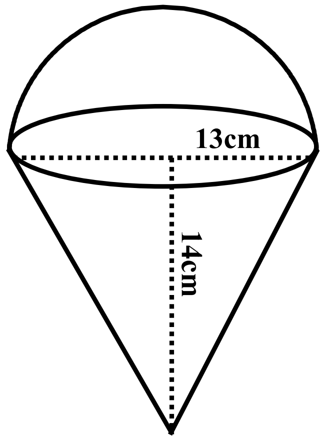
3. The diagram shows a piece of cheese cut in the form of a triangular prism. The dimensions are indicated on the diagram.
   (a) Calculate the volume of the cheese.
   (b) Calculate the surface area of the wrapping paper around the cheese.



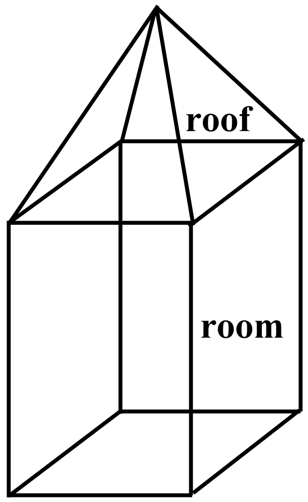
4. A bucket is in the shape of a frustum. If the diameters of the circles are 20cm and 12cm and the depth is 30cm, calculate the volume of the bucket.



5. Consider the following toy made from a cone with height 14cm and a hemisphere (half of a sphere) with a radius of 13cm. Calculate the volume and total exterior surface area of the toy.

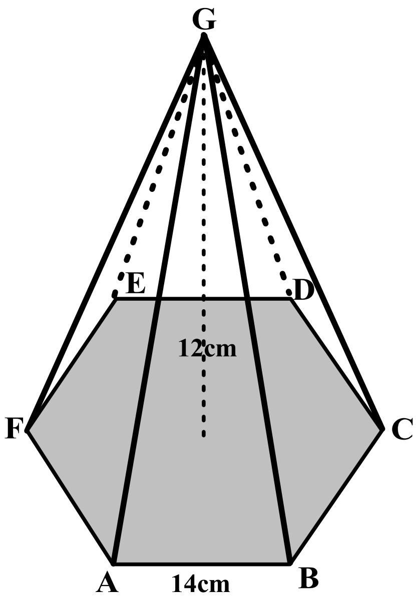


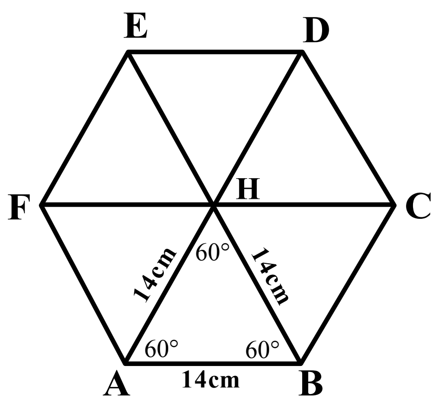
6. A barn is constructed as follows:
The room space is constructed as a right rectangular prism with a square base. The length of one side of the base of the prism is equal to 15 metres. The height of the wall of the room is 20 metres. The roof is constructed in the form of a right triangular pyramid with a height of 10 metres.
The base of the roof is open. Calculate:
(a) the slant height of the triangular face of the pyramid.
(b) the area of one of the triangular faces.
(c) the total exterior surface area of the barn.
(d) the volume of the barn.



CHALLENGE

Calculate the volume and surface area of the given hexagonal pyramid correct to one decimal place.





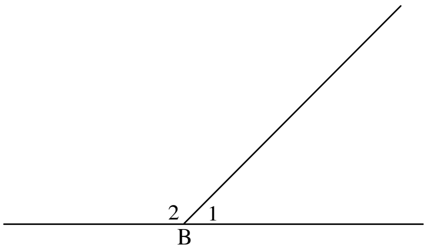
CHAPTER 8 – EUCLIDEAN GEOMETRY

REVISION OF EARLIER CONCEPTS (GRADE 8-10)

LINES AND ANGLES

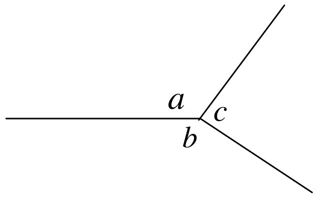
Adjacent supplementary angles

In the diagram, $\hat{B}_1 + \hat{B}_2 = 180°$



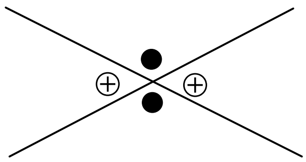
Angles round a point

In the diagram, $a + b + c = 360°$



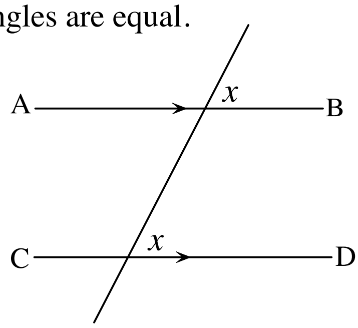
Vertically opposite angles

Vertically opposite angles are equal.



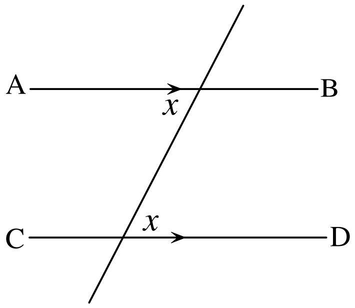
Corresponding angles

If AB||CD, then the corresponding angles are equal.



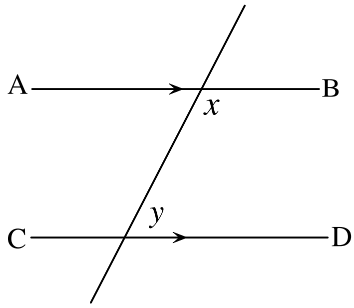
Alternate angles

If AB||CD, then the alternate angles are equal.



Co-interior angles

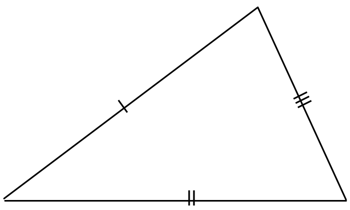
If AB||CD, then the co-interior angles add up to $180°$, i.e. $x + y = 180°$



TRIANGLES

Scalene Triangle

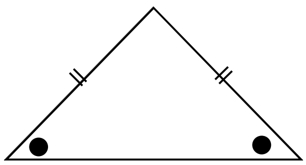
No sides are equal in length



Isosceles Triangle

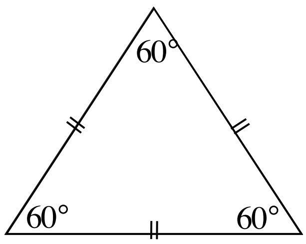
Two sides are equal

Base angles are equal



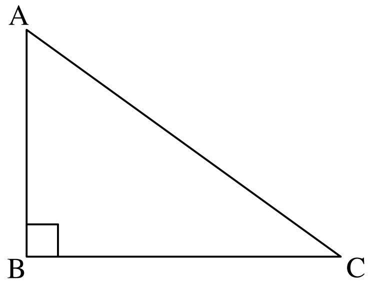
Equilateral Triangle

All three sides are equal
All three interior angles are equal

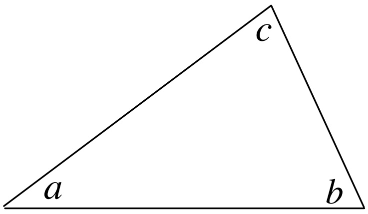


Right-angled triangle

One interior angle is $90°$

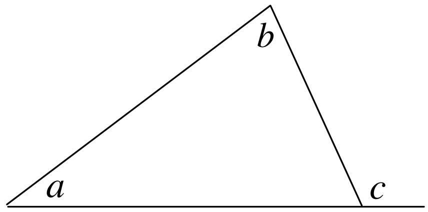


Sum of the angles of a triangle



$a + b + c = 180°$

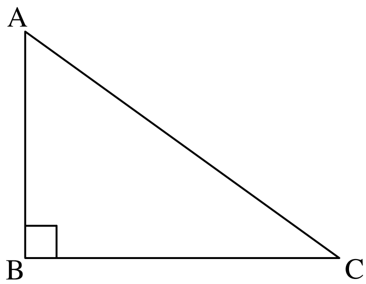
Exterior angle of a triangle



$c = a + b$

The Theorem of Pythagoras

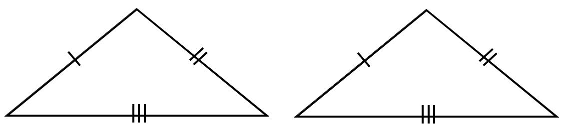
$AC^2 = AB^2 + BC^2$
or
$AB^2 = AC^2 - BC^2$
or
$BC^2 = AC^2 - AB^2$



<u>Congruency of triangles (four conditions)</u>

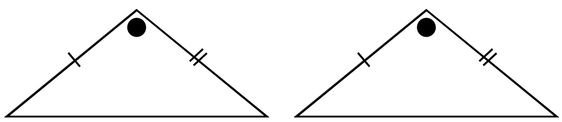
Condition 1

Two triangles are congruent if three sides of one triangle are equal in length to the three sides of the other triangle.



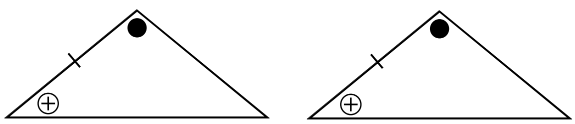
Condition 2

Two triangles are congruent if two sides and the included angle are equal to two sides and the included angle of the other triangle.



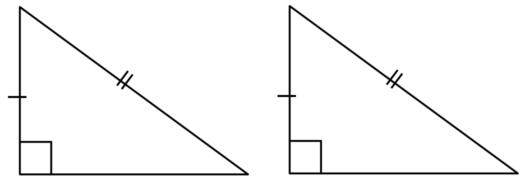
Condition 3

Two triangles are congruent if two angles and one side of a triangle are equal to two angles and a corresponding side of the other triangle.

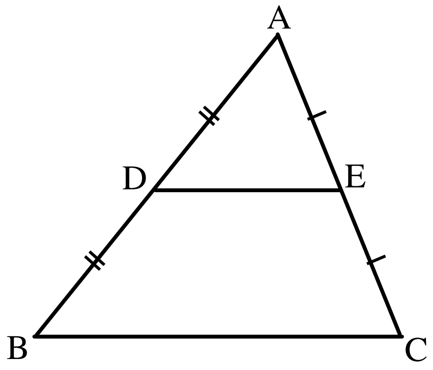


Condition 4

Two right-angled triangles are congruent if the hypotenuse and a side of the one triangle is equal to the hypotenuse and a side of the other triangle.

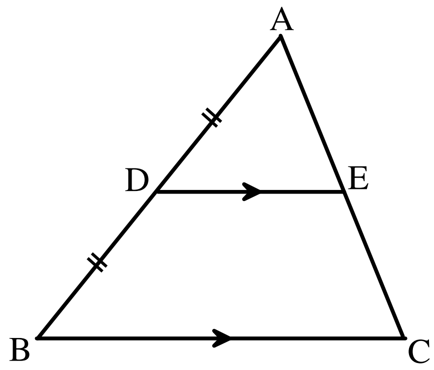


The Midpoint Theorem



If $AD = DB$ and $AE = EC$,

then $DE \parallel BC$ and $DE = \frac{1}{2}BC$



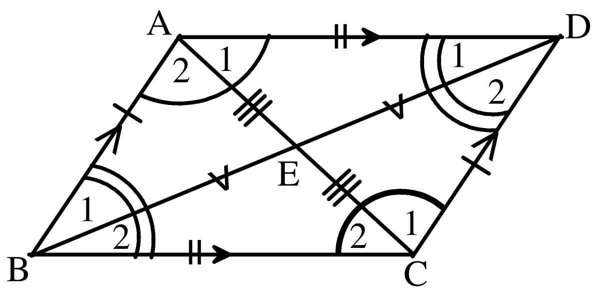
If $AD = DB$ and $DE \parallel BC$,

then $AE = EC$ and $DE = \frac{1}{2}BC$.

<u>PROPERTIES OF QUADRILATERALS</u>

<u>Parallelograms</u>

If ABCD is a parallelogram, you may assume the following properties:



$AD \parallel BC$; $AB \parallel DC$

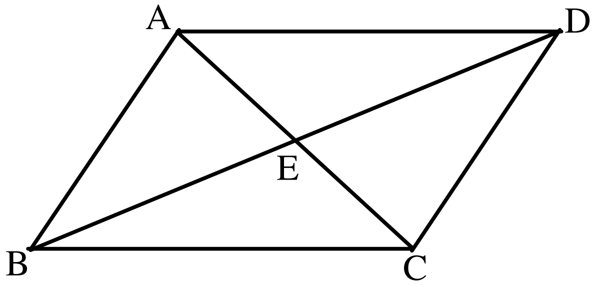
$AD = BC$; $AB = DC$

$AE = EC$; $BE = ED$

$\hat{D}_1 = \hat{B}_2$; $\hat{D}_2 = \hat{B}_1$; $\hat{C}_1 = \hat{A}_2$; $\hat{C}_2 = \hat{A}_1$

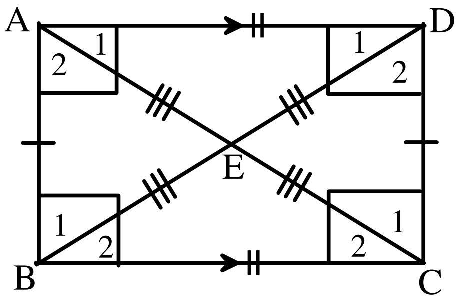
$\hat{A} = \hat{C}$; $\hat{B} = \hat{D}$

In order to prove that a quadrilateral is a parallelogram, you will need to prove at least one of the following:



$AD \parallel BC$ and $AB \parallel DC$	Opp sides $\parallel$
$AD = BC$ and $AB = DC$	Opp sides =
$AE = EC$ and $BE = ED$	Diagonals bisect
$\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$	Opp angles =
$AB \parallel DC$ and $AB = DC$	One pair opp sides = and $\parallel$
$AD \parallel BC$ and $AD = BC$	One pair opp sides = and $\parallel$

Rectangle
If ABCD is a rectangle, you may assume the following properties:

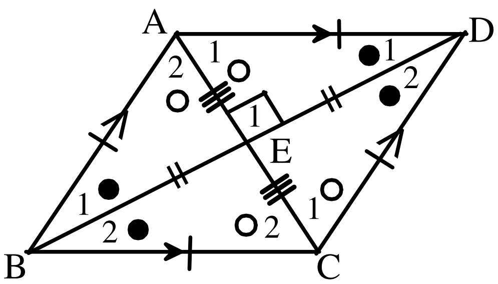


$AD \parallel BC$; $AB \parallel DC$
$AD = BC$; $AB = DC$
$AE = EC = BE = ED$
$\hat{D}_1 = \hat{B}_2$; $\hat{D}_2 = \hat{B}_1$; $\hat{C}_1 = \hat{A}_2$; $\hat{C}_2 = \hat{A}_1$
$\hat{A} = \hat{C} = \hat{B} = \hat{D} = 90°$

In order to prove that a quadrilateral is a rectangle, you will need to prove one of the following:

(a) The quadrilateral is a parallelogram with at least one interior angle equal to $90°$.
(b) The diagonals of the quadrilateral are equal in length and bisect each other.

Rhombus
If ABCD is a rhombus, you may assume the following properties:

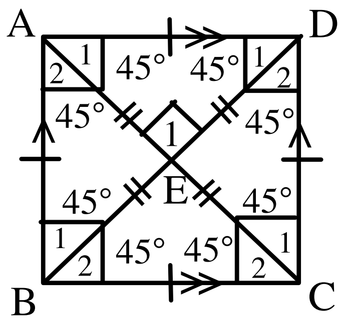


$AD \parallel BC$; $AB \parallel DC$
$AD = BC = AB = DC$
$AE = EC$; $BE = ED$
$\hat{D}_1 = \hat{D}_2 = \hat{B}_1 = \hat{B}_2$
$\hat{A}_1 = \hat{A}_2 = \hat{C}_1 = \hat{C}_2$; $\hat{A} = \hat{C}$; $\hat{B} = \hat{D}$
$\hat{E}_1 = 90°$; $AC \perp BD$

In order to prove that a quadrilateral is a rhombus, you will need to prove one of the following:

(a) The quadrilateral is a parallelogram with a pair of adjacent sides equal
(b) The quadrilateral is a parallelogram in which the diagonals bisect at right angles.

Square
If ABCD is a square, you may assume the following properties:



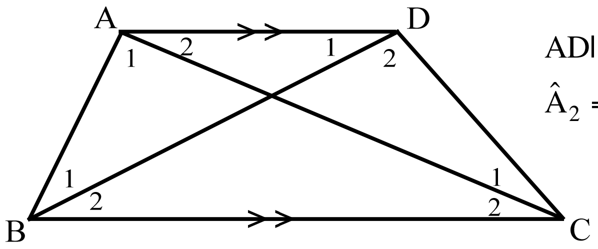
$AD \parallel BC$; $AB \parallel DC$
$AD = BC = AB = DC$
$AE = EC = BE = ED$
$\hat{D}_1 = \hat{D}_2 = \hat{B}_1 = \hat{B}_2 = \hat{A}_1 = \hat{A}_2 = \hat{C}_1 = \hat{C}_2 = 45°$
$\hat{A} = \hat{C} = \hat{B} = \hat{D} = 90°$
$\hat{E}_1 = 90°$; $AC \perp BD$

In order to prove that a quadrilateral is a square, you will need to prove one of the following:

(a) The quadrilateral is a parallelogram with an interior right angle and a pair of adjacent sides equal.
(b) The quadrilateral is a rhombus with an interior right angle
(c) The quadrilateral is a rhombus with equal diagonals.

Trapezium

If ABCD is a trapezium, you may assume the following properties:



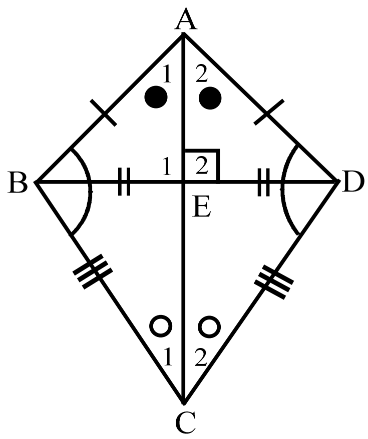
AD||BC

$\hat{A}_2 = \hat{C}_2$; $\hat{D}_1 = \hat{B}_2$

In order to prove that a quadrilateral is a trapezium, you will need to prove that AD||BC.

Kite

If ABCD is a kite, you may assume the following properties:



AB = AD

BC = DC

BE = ED

$\hat{A}_1 = \hat{A}_2$

$\hat{C}_1 = \hat{C}_2$

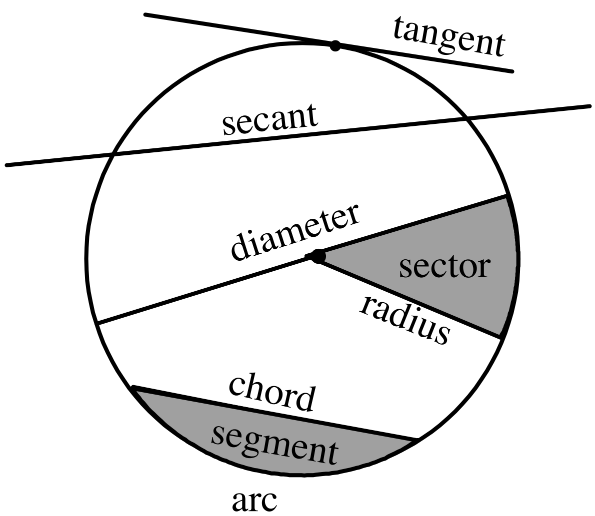
$\hat{B} = \hat{D}$

$\hat{E}_2 = 90°$

AC ⊥ BD

In order to prove that a quadrilateral is a kite, you will need to prove that the pairs of adjacent sides are equal in length.

BASIC CIRCLE TERMINOLOGY



Radius:
A line from the centre to any point on the circumference of the circle.
Diameter:
A line passing through the centre of the circle. It is double the length of the radius.
Chord:
A line with end-points on the circumference.
Tangent:
A line touching the circle at only one point.
Secant:
A line passing through two points on the circle.

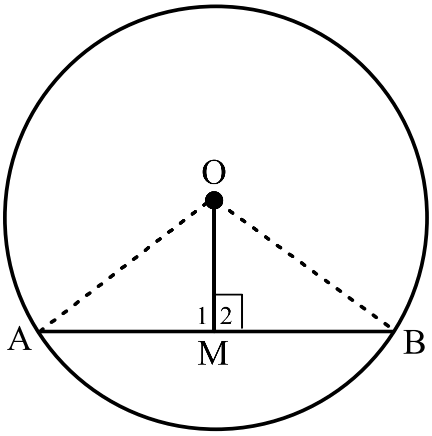
THEOREM 1(A)

The perpendicular drawn from the centre of a circle to a chord bisects the chord.

Theorem 1 in a nutshell according to the given circle sketched below:
If $OM \perp AB$ (which means that $\hat{M}_1 = \hat{M}_2 = 90°$) then $AM = MB$

Given: Circle with centre O with $OM \perp AB$. AB is a chord

Required to prove: $AM = MB$.



Proof

Join OA and OB.

In ΔOAM and ΔOBM:

(a) $OA = OB$ ….. radii

(b) $\hat{M}_1 = \hat{M}_2 = 90°$ ….. given

(c) $OM = OM$ ….. common

$\therefore \Delta OAM \equiv \Delta OBM$ ….. RHS

$\therefore AM = MB$

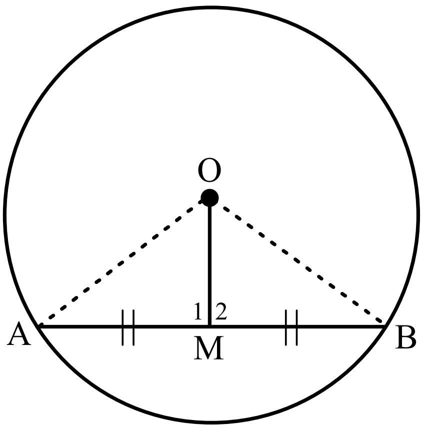
CONVERSE THEOREM 1(A)

The line segment joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Converse of theorem 1 in a nutshell according to the given circle sketched below: If $AM = MB$ then $OM \perp AB$ (which means that $\hat{M}_1 = \hat{M}_2 = 90°$)

Given: Circle with centre O. M is a point on chord AB such that $AM = MB$.

Required to prove: $OM \perp AB$ (which means that $\hat{M}_1 = \hat{M}_2 = 90°$)



Proof

Join OA and OB

In ΔOAM and ΔOBM:

(a) $OA = OB$ ….. radii

(b) $AM = BM$ ….. given

(c) $OM = OM$ ….. common

$\therefore \Delta OAM \equiv \Delta OBM$ ….. SSS

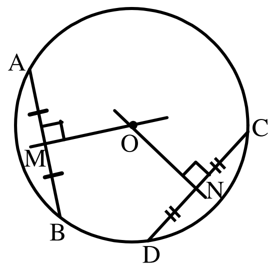
$\therefore \hat{M}_1 = \hat{M}_2$

But AMB is a straight line

$\therefore \hat{M}_1 = \hat{M}_2 = 90°$ ….. Adjacent supplementary $\angle s$

Definition

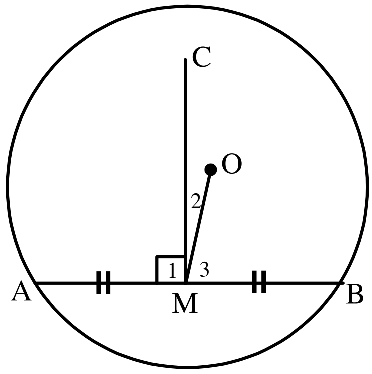
The perpendicular bisector of a line is a line that bisects the given line at right angles. In the diagram, OM is the perpendicular bisector of AB and ON is the perpendicular bisector of CD.



THEOREM 1(B)

The perpendicular bisector of a chord passes through the centre of the circle.

Given: Chord AMB of circle with centre O. CM is the perpendicular bisector of chord AMB.
Required to prove: O lies on CM



Proof

Let's assume that the centre O doesn't lie on the perpendicular bisector CM.

$\hat{M}_1 = 90°$ Given

But $\hat{M}_3 = 90°$ Line from centre O to midpoint of chord AMB.

This means that:

$\hat{M}_1 + \hat{M}_3 = 180°$

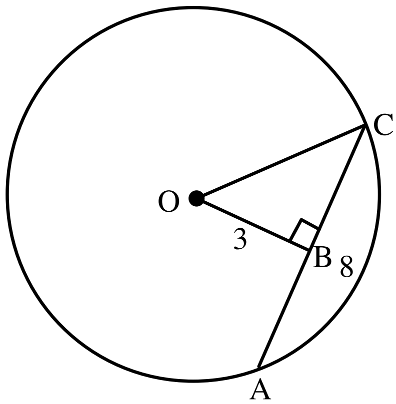
which is impossible since $\hat{M}_1 + \hat{M}_2 + \hat{M}_3 = 180°$ adjacent supplementary angles

$\therefore \hat{M}_2 = 0°$

$\therefore$ O lies on CM

EXAMPLE 1

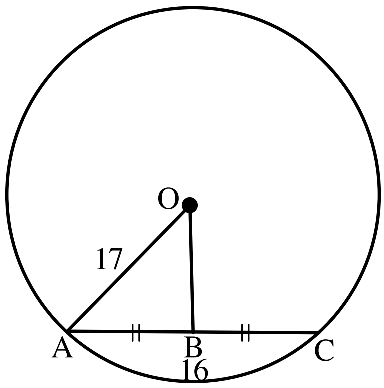
O is the centre. AC = 8, OB = 3 and $OB \perp AC$.
Calculate: (a) AB (b) OC



Statement	Reason
(a) AB = BC = 4	Perp from centre to chord.
(b) BC = 4	Proved
$OC^2 = (3)^2 + (4)^2$	
$\therefore OC^2 = 25$	
$\therefore OC = 5$	

EXAMPLE 2

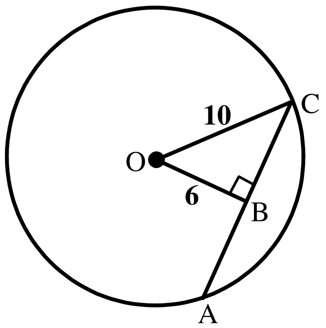
O is the centre. AC = 16, AB = BC and OA = 17.
Calculate the length of OB.



Statement	Reason
AB = 8	Given
$O\hat{B}A = 90°$	Line from centre to midpoint of chord
$OB^2 = (17)^2 - (8)^2$	Pythagoras
$\therefore OB^2 = 225$	
$\therefore OB = 15$	

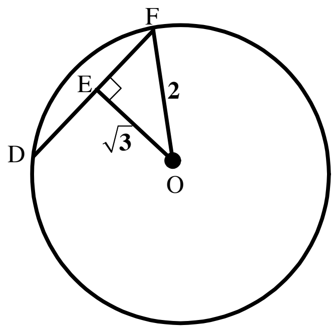
EXERCISE 1

(a) O is the centre. OB = 6, OC = 10 and $OB \perp AC$. Calculate the length of AC.



(b) O is the centre. EO = $\sqrt{3}$, OF = 2 and $OE \perp DF$. Calculate the length of DE.

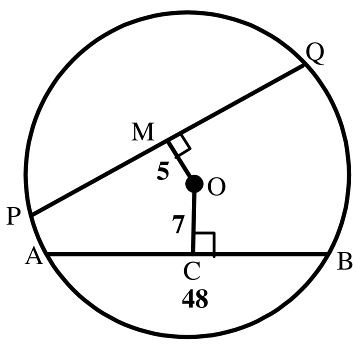
.



(c) O is the centre of the circle. PQ and AB are both chords of the circle with $OM \perp PQ$ and $OC \perp AB$. OM = 5cm, OC = 7cm and AB = 48cm. Calculate the length of:

(1) the radius of the circle.

(2) PQ.

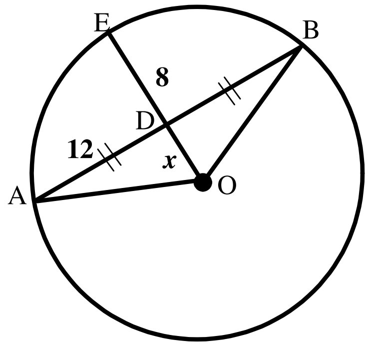


Statement	Reason

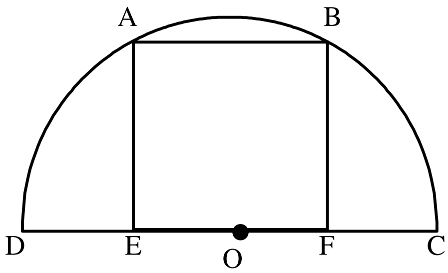
(d) AB is a chord of circle centre O. OE bisects AB. AD = 12cm, ED = 8cm and OD = x.

(1) Determine the radius OB in terms of x.

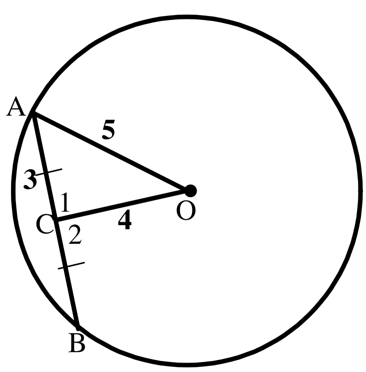
(2) Hence, calculate the length of the radius OB.



(e) The radius of the semi-circle centre O is 5 cm. A square is fitted into the semi-circle as shown in the diagram. Calculate the area of the square. (Hint: Let the length of the square equal x.)

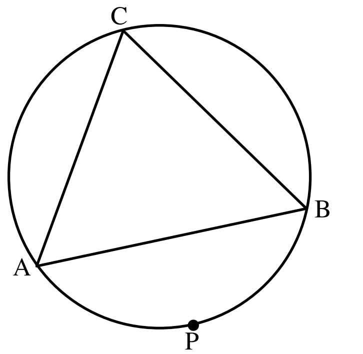


(f) AB is a chord of the circle. AC = CB, and the length of the radius is 5 units. AC = 3 units and OC = 4 units. Show that $OC \perp AB$ and explain why OC passes through the centre O.

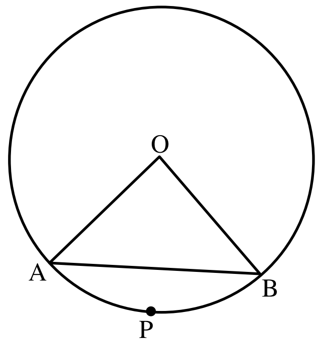


Statement	**Reason**

Subtended Angles



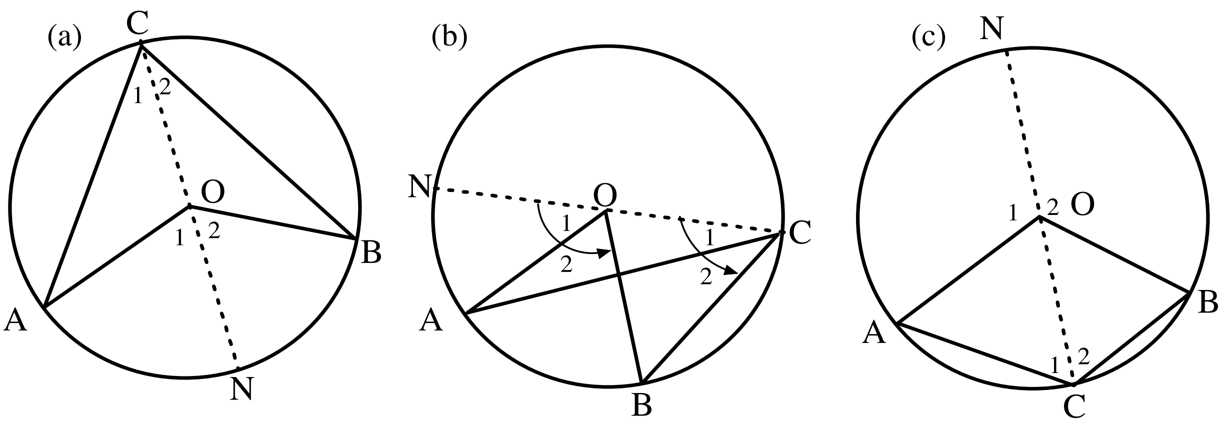
Arc APB or chord AB **subtends**
$\hat{C}$ at the circumference



Arc APB or chord AB **subtends**
$A\hat{O}B$ at the centre of the circle

THEOREM 2

The angle which an arc of a circle subtends at the centre of a circle is twice the angle it subtends at the circumference of the circle. ($\angle$ at the centre = $2\times\angle$ at circumference)



Given: Circle with centre O and A, B and C are all points on the circumference of the circle.

Required to prove: $A\hat{O}B = 2A\hat{C}B$

For diagrams (a) and (c)

Proof:

Join CO and produce to N.

$\hat{O}_1 = \hat{C}_1 + \hat{A}$ ..... Ext $\angle$ of ΔOAC

But $\hat{C}_1 = \hat{A}$ ..... $OA = OC$, Radii

$\therefore \hat{O}_1 = 2\hat{C}_1$

Similarly, in ΔOCB $\hat{O}_2 = 2\hat{C}_2$

$\therefore \hat{O}_1 + \hat{O}_2 = 2\hat{C}_1 + 2\hat{C}_2$

$\therefore \hat{O}_1 + \hat{O}_2 = 2\left(\hat{C}_1 + \hat{C}_2\right)$

$\therefore A\hat{O}B = 2A\hat{C}B$

For diagram (b)

Proof:

Join CO and produce to N.

$\hat{O}_1 = \hat{C}_1 + \hat{A}$ ..... Ext $\angle$ of ΔOAC

But $\hat{C}_1 = \hat{A}$ ..... $OA = OC$, Radii

$\therefore \hat{O}_1 = 2\hat{C}_1$

Similarly, in ΔOCB $\hat{O}_2 = 2\hat{C}_2$

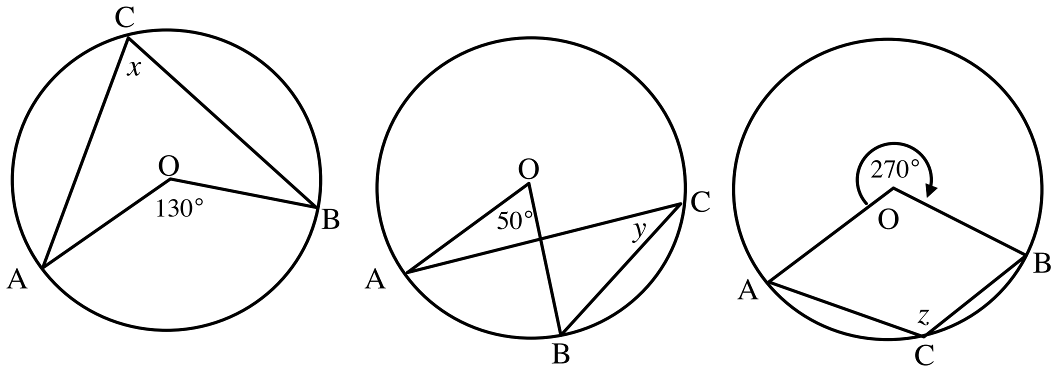
$\therefore \hat{O}_2 - \hat{O}_1 = 2\hat{C}_2 - 2\hat{C}_1$

$\therefore \hat{O}_2 - \hat{O}_1 = 2\left(\hat{C}_2 - \hat{C}_1\right)$

$\therefore A\hat{O}B = 2A\hat{C}B$

EXAMPLE 3

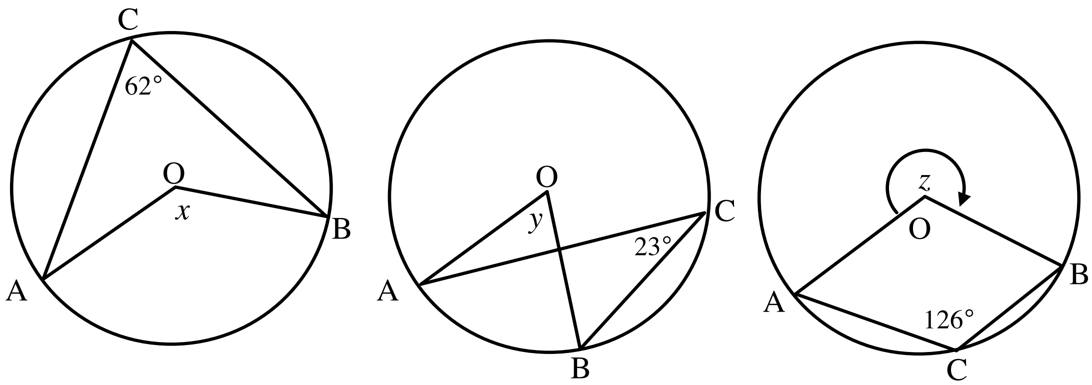
O is the centre of each circle. Calculate x, y and z.



Statement	Reason
$x = 65°$	∠ at centre = 2×∠ at circumference
$y = 25°$	∠ at centre = 2×∠ at circumference
$z = 135°$	∠ at centre = 2×∠ at circumference

EXAMPLE 4

O is the centre of each circle. Calculate x, y and z.

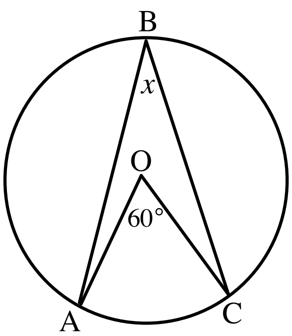


Statement	Reason
$x = 124°$	∠ at centre = 2×∠ at circumference
$y = 46°$	∠ at centre = 2×∠ at circumference
$z = 252°$	∠ at centre = 2×∠ at circumference

EXERCISE 2

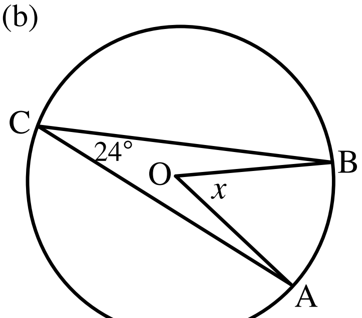
Calculate the value of the unknown variables. O is the centre in each case.

(a)

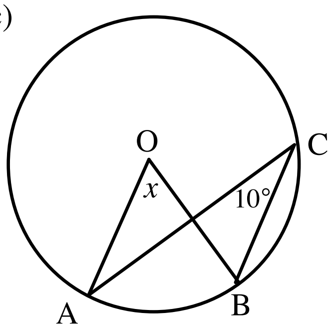


Statement	Reason

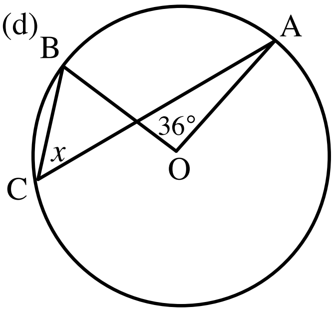
(b)



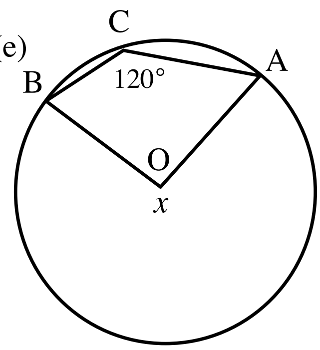
(c)



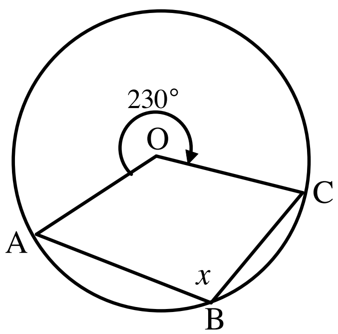
(d)



(e)

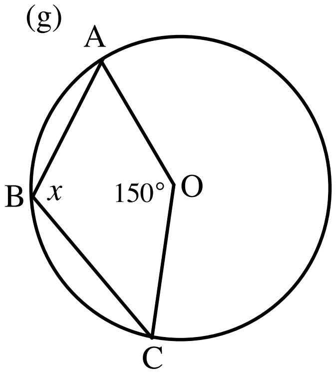


(f)

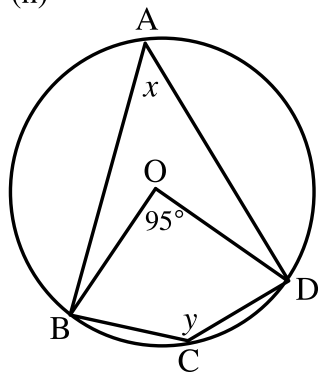


Statement	Reason

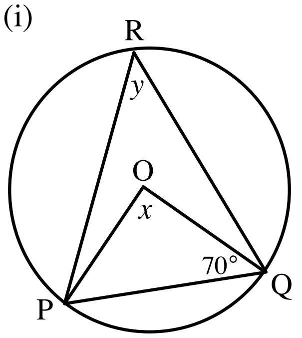
(g)



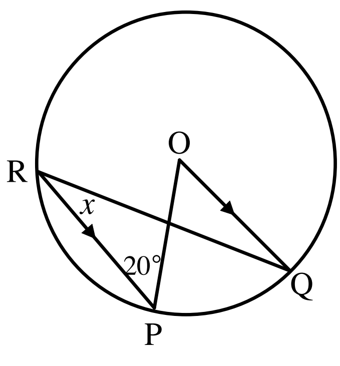
(h)



(i)



(j)



Statement	**Reason**

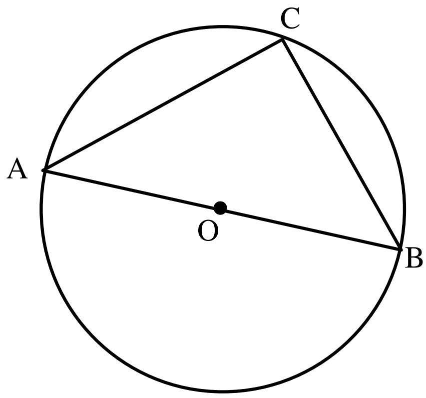
THEOREM 3

The angle subtended at the circle by a diameter is a right angle. We say that the angle in a semi-circle is $90°$. ($\angle$ in a semi-circle)

Theorem 3 in a nutshell according to the given circle sketched below: If AB is a diameter (a chord passing through the centre) then $\hat{C} = 90°$

Given: Circle with diameter AOB

Required to prove: $\hat{C} = 90°$.



Proof

$A\hat{O}B = 2\hat{C}$ ….. $\angle$ at the centre = $2 \times \angle$ at circumference

$A\hat{O}B = 180°$ ….. straight line

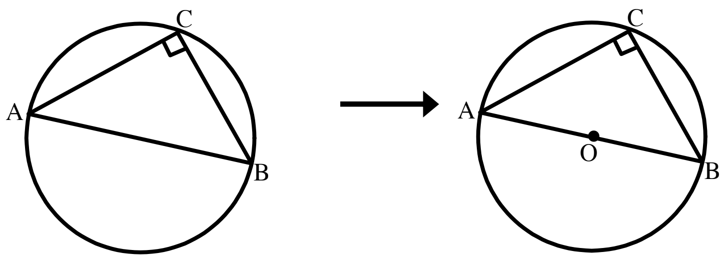
$\therefore 180° = 2\hat{C}$

$\therefore \hat{C} = 90°$

CONVERSE OF THEOREM 3

If the angle subtended by a chord at a point on the circle is $90°$, then the chord is a diameter. (chord subtends $90°$)

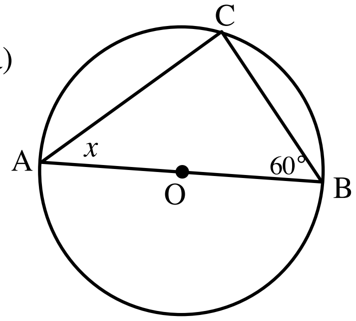
Converse of Theorem 3 in a nutshell according to the given circle sketched below: If $\hat{C} = 90°$ then chord AB is a diameter (passes through the centre)



EXAMPLE 5

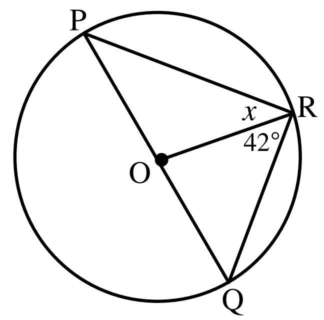
Calculate the value of the unknown variables. O is the centre in each case.

(a)



Statement	Reason
$\hat{C} = 90°$	$\angle$ in a semi-circle
$x = 30°$	Sum of the angles of Δ

(b)

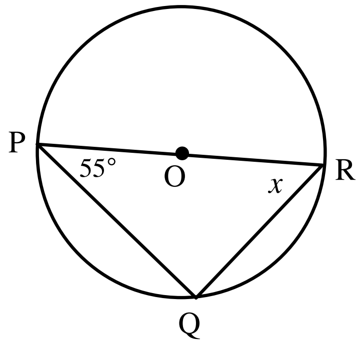


Statement	Reason
$\hat{R} = 90°$	∠ in a semi-circle
$\therefore x = 48°$	$O\hat{R}Q = 42°$

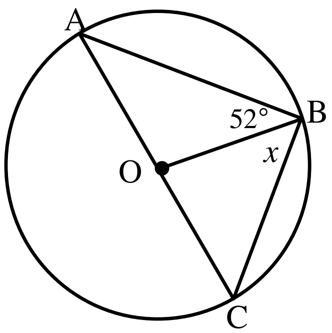
EXERCISE 3

Calculate the value of the unknown variables. O is the centre.

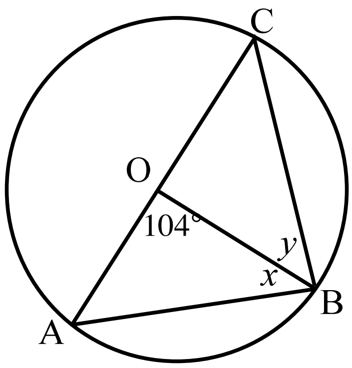
(a)



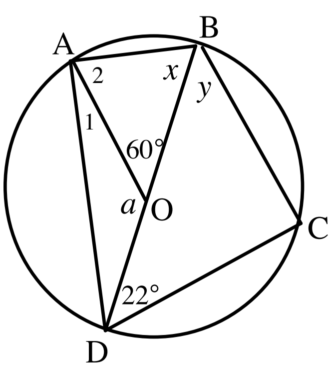
(b)



(c)



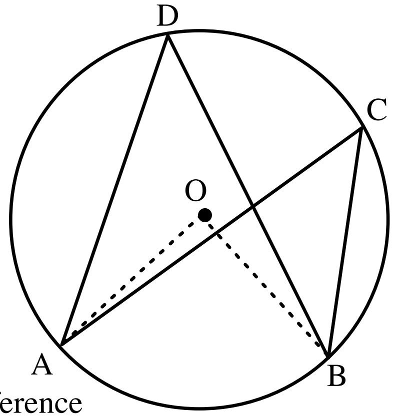
(d)



Statement	Reason

THEOREM 4

An arc or chord of a circle subtends equal angles at the circumference of the circle. We say that the angles in the same segment of the circle are equal.
(arc / chord subtends equal $\angle$ s)



Given: Circle with centre O and A, B, C and D are all points on the circumference of the circle.

Required to prove: $\hat{A} = \hat{B}$ and $\hat{C} = \hat{D}$.

Proof

Join OA and OB

$A\hat{O}B = 2\hat{C}$ — $\angle$ at the centre = $2 \times \angle$ at circumference

and $A\hat{O}B = 2\hat{D}$ — $\angle$ at the centre = $2 \times \angle$ at circumference

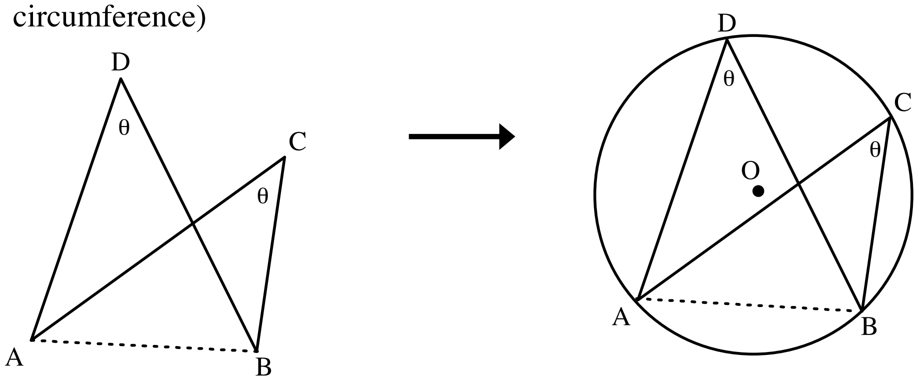
$\therefore 2\hat{C} = 2\hat{D}$

$\therefore \hat{C} = \hat{D}$

Similarly by joining OD and OC, $\hat{A} = \hat{B}$

CONVERSE OF THEOREM 4

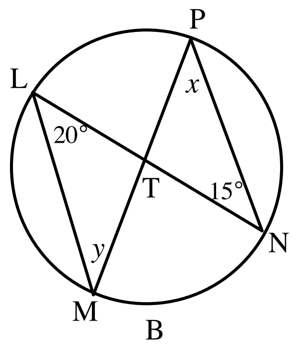
If a line segment joining two points subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic (lie on the circumference)



EXAMPLE 6

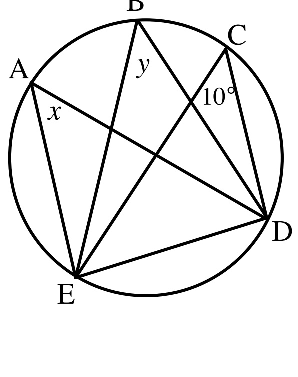
Calculate the value of the unknown angles.

(a)



Statement	Reason
$x = 20°$ $y = 15°$	Arc MN subtends $= \angle$ s Arc LP subtends $= \angle$ s

(b)

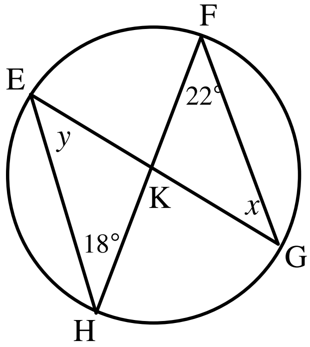


Statement	Reason
$y = 10°$ $x = 10°$	Arc ED subtends $= \angle$ s Arc ED subtends $= \angle$ s

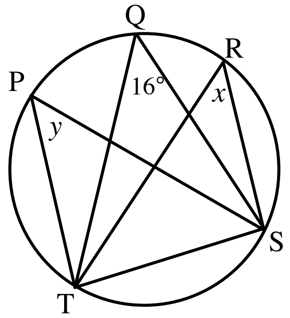
EXERCISE 4

Calculate the value of the unknown angles. O is the centre.

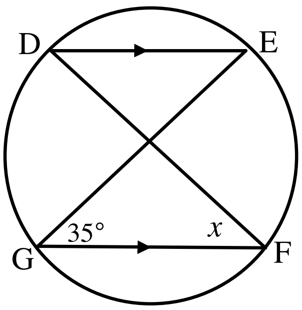
(a)



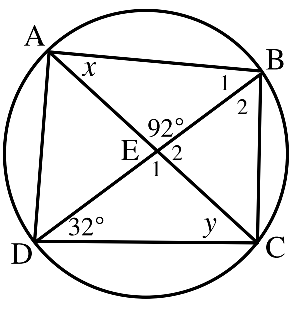
(b)



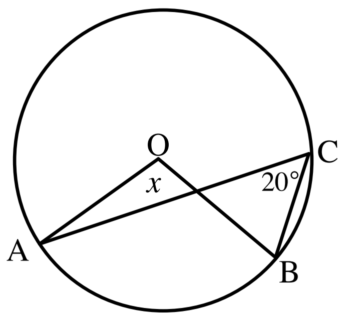
(c)



(d)

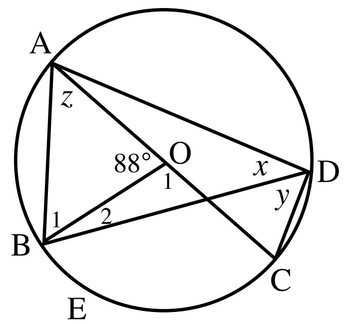


(e)

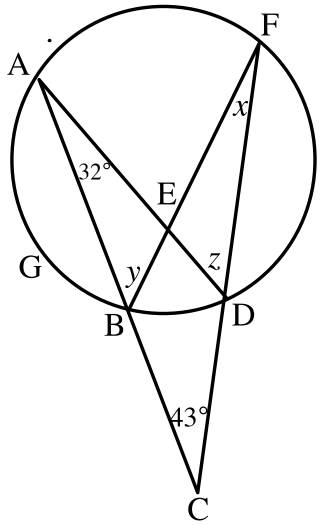


Statement	Reason

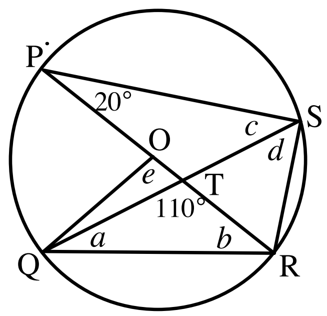
(f)



(g)



(h)



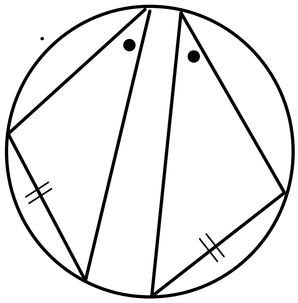
Statement	Reason

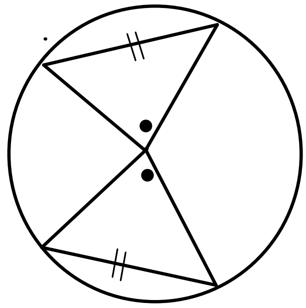
CORROLLARIES ON THEOREM 4

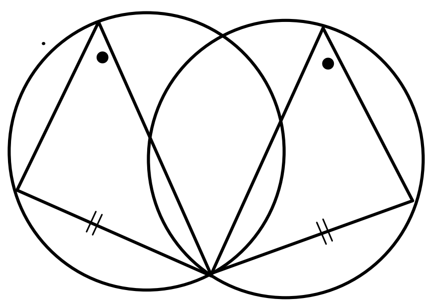
1. Equal chords subtend equal angles at the circumference

2. Equal chords subtend equal angles at the centre

3. Equal chords of equal circles subtend equal angles at the circumference.



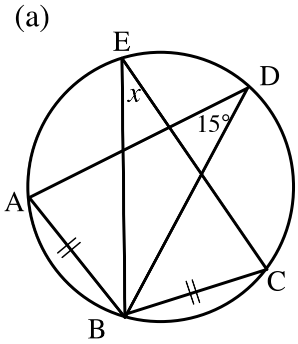




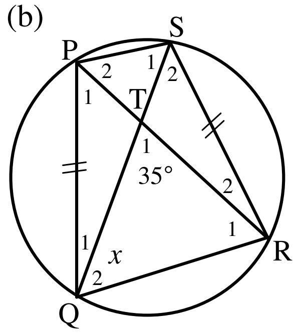
EXERCISE 5

1. Calculate the unknown angles.

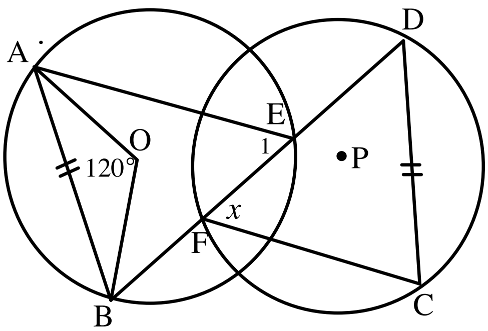
(a)



(b)



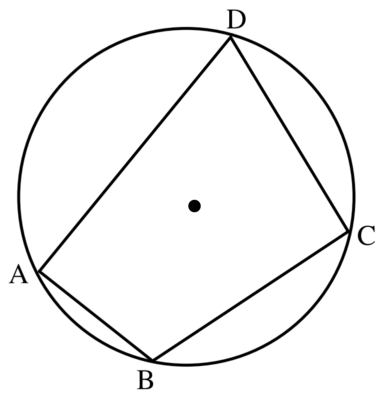
2. Two equal circles are given. O is the centre of circle ABE. Calculate the value of x. Reasons must be given.



Statement	Reason

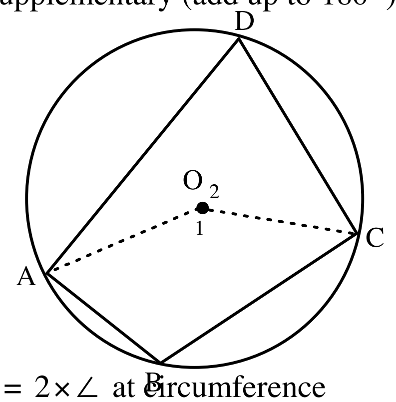
CYCLIC QUADRILATERALS

A quadrilateral whose vertices lie on the circumference of a circle is referred to as a **cyclic quadrilateral**. ABCD is a cyclic quadrilateral because A, B, C and D are concyclic.



THEOREM 5

The opposite angles of a cyclic quadrilateral are supplementary (add up to $180°$)
(opp $\angle s$ cyclic quad)



Given: A, B, C and D are points that lie on the circumference of the circle (ABCD is a cyclic quadrilateral)

Required to prove: $\hat{A}+\hat{C}=180°$ and $\hat{B}+\hat{D}=180°$

Proof

Join AO and OC.

$\hat{O}_1 = 2\hat{D}$ ….. $\angle$ at centre $= 2\times\angle$ at circumference

$\hat{O}_2 = 2\hat{B}$ ….. $\angle$ at centre $= 2\times\angle$ at circumference

$\hat{O}_1 + \hat{O}_2 = 2\hat{D}+2\hat{B}$

And $\hat{O}_1 + \hat{O}_2 = 360°$ ….. $\angle's$ at a point

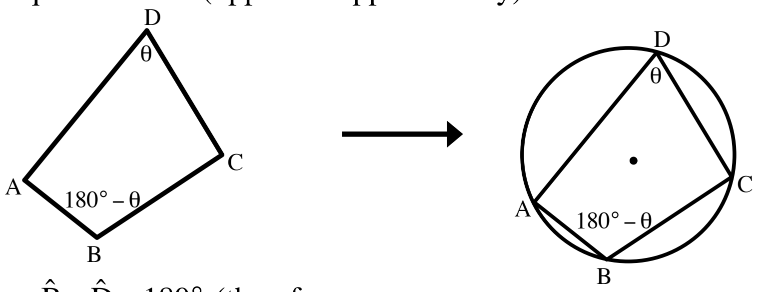
$\therefore 360° = 2\left(\hat{D}+\hat{B}\right)$

$\therefore 180° = \hat{D}+\hat{B}$

Similarly, by joining BO and DO, it can be proven that $\hat{A}+\hat{C}=180°$

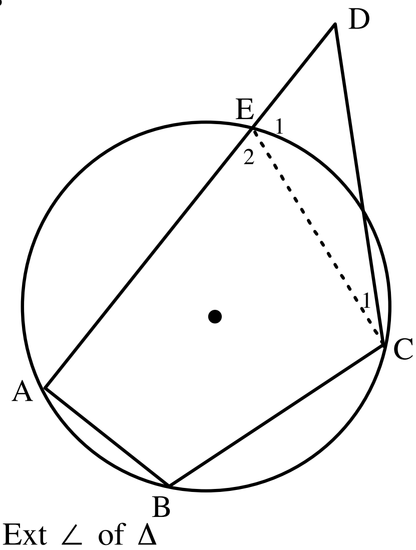
CONVERSE OF THEOREM 5

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral. (opp $\angle s$ supplementary)



Given: $\hat{B}+\hat{D}=180°$ (therefore $\hat{A}+\hat{C}=180°$ …interior angles of a quad add up to $360°$)

Required to prove: ABCD is a cyclic quadrilateral



Proof

Draw a circle through point A, B and C.
Assume that the circle doesn't pass through D, but cuts AD at E.

$\hat{E}_2+\hat{B}=180°$ …. opp $\angle s$ cyclic quad

AND $\hat{D}+\hat{B}=180°$ …. given

$\therefore \hat{E}_2=\hat{D}$

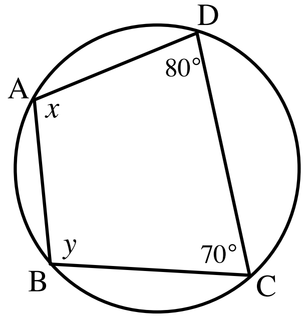
But this is impossible since $\hat{E}_2=\hat{D}+\hat{C}_1$ ….. Ext $\angle$ of Δ

$\therefore$ E and D must be the same point
$\therefore$ ABCD is a cyclic quad

EXAMPLE 7

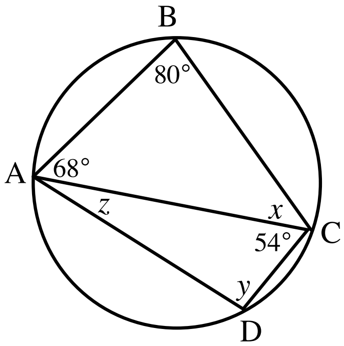
Calculate the value of the unknown angles.

(a)



Statement	Reason
$x = 110°$ $y = 100°$	Opp $\angle$ s of cyclic quad Opp $\angle$ s of cyclic quad

(b)

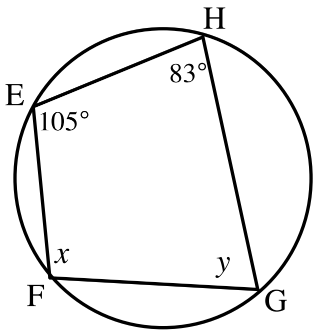


Statement	Reason
$x = 32°$ $y = 100°$ $z = 26°$	Sum of the angles of Δ Opp $\angle$ s of cyclic quad Sum of the angles of Δ

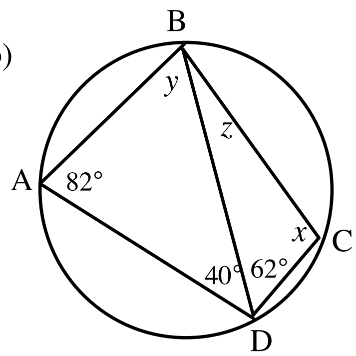
EXERCISE 6

Calculate the value of the unknown angles. O is the centre.

(a)

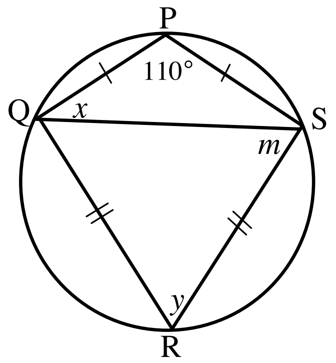


(b)

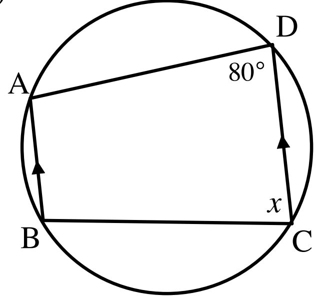


Statement	Reason

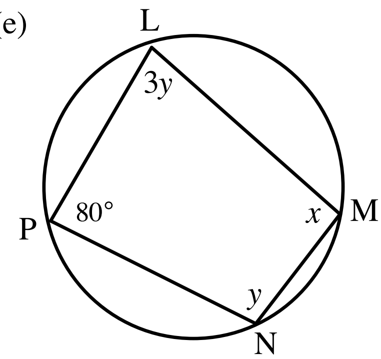
(c)



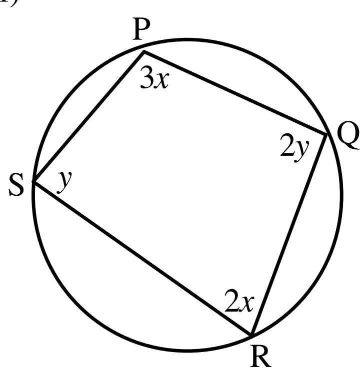
(d)



(e)

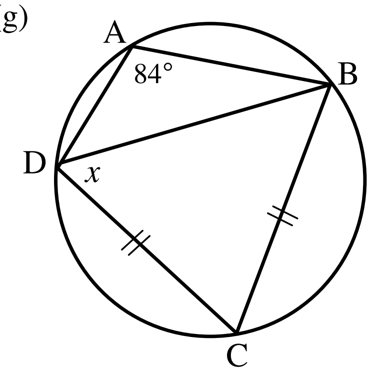


(f)

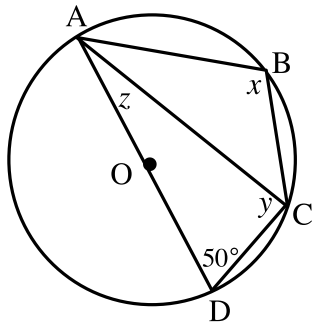


Statement	**Reason**

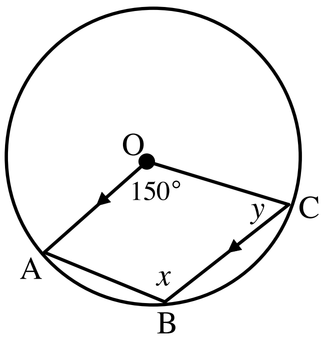
(g)



(h)



(i)



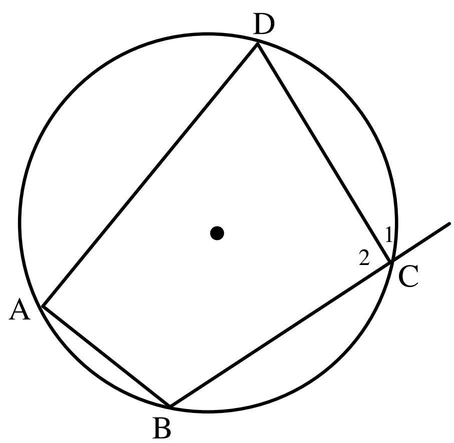
Statement	**Reason**

THEOREM 6

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
(ext $\angle$ of cyclic quad)

Given: A, B, C and D are points that lie on the circumference of the circle (ABCD is a cyclic quadrilateral)

Required to prove: $\hat{C}_1 = \hat{A}$



Proof

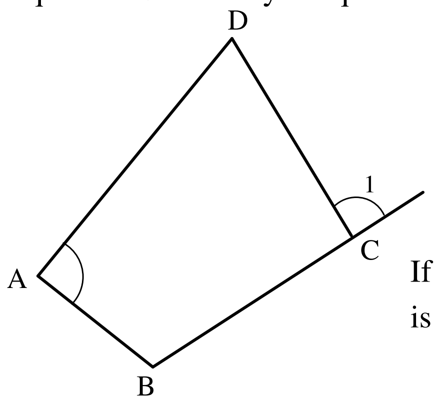
$\hat{C}_2 + \hat{A} = 180°$ ….. opp $\angle s$ of cyclic quad

$\hat{C}_2 + \hat{C}_1 = 180°$ ….. $\angle s$ on a straight line

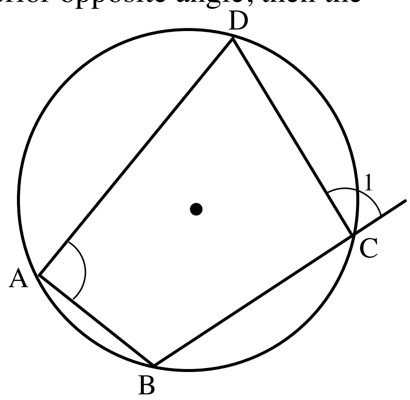
$\therefore \hat{C}_1 = \hat{A}$

CONVERSE OF THEOREM 6

If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is a cyclic quadrilateral.

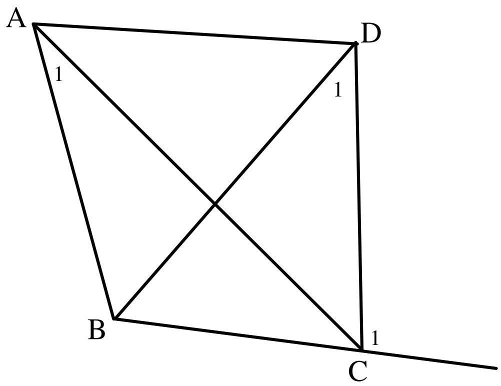


If $\hat{C}_1 = \hat{A}$ then ABCD is a cyclic quadrilateral



Important Summary (Strategies to prove that a quadrilateral is cyclic)

ABCD is a quadrilateral. Write down three conditions which would make ABCD a cyclic quadrilateral.



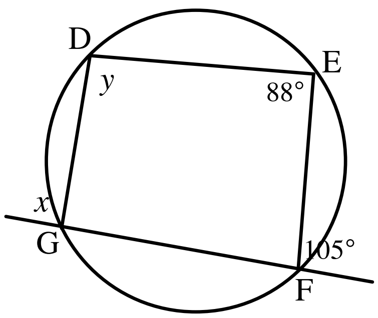
Condition 1: $B\hat{A}D + B\hat{C}D = 180°$
$A\hat{B}C + A\hat{D}C = 180°$

Condition 2: $\hat{C}_1 = B\hat{A}D$

Condition 3: $\hat{A}_1 = \hat{D}_1$

EXAMPLE 8

Calculate the value of the unknown angles. O is the centre.

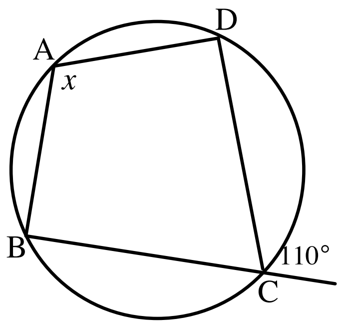


Statement	Reason
$x = 88°$	Ext $\angle$ of cyclic quad
$y = 105°$	Ext $\angle$ of cyclic quad

EXERCISE 7

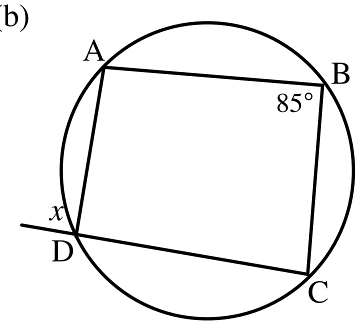
1. Calculate the value of the unknown angles. O is the centre.

(a)

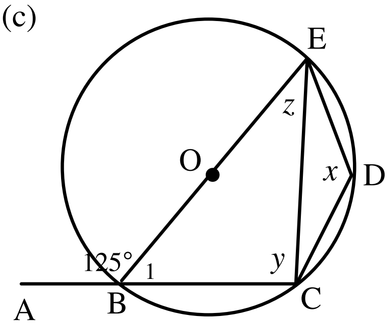


Statement	Reason

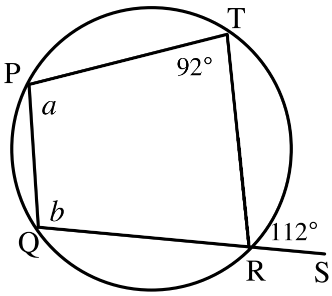
(b)



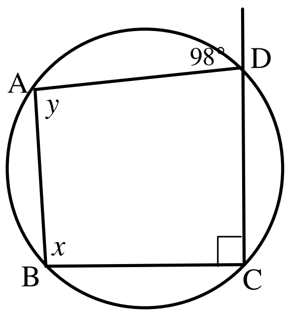
(c)



(d)

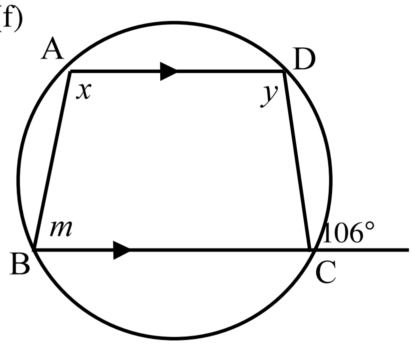


(e)

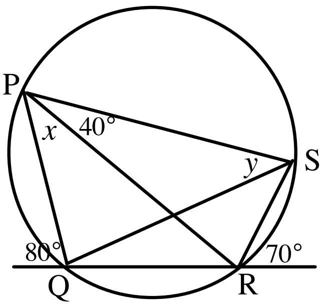


Statement	**Reason**

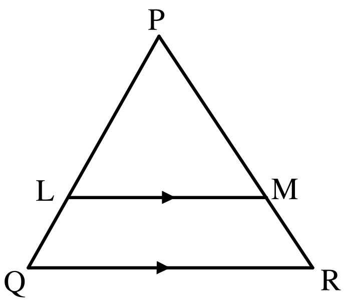
(f)



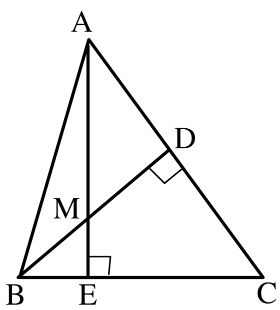
(g)



2(a) Show that LMRQ is a cyclic quadrilateral if PQ = PR and LM || QR



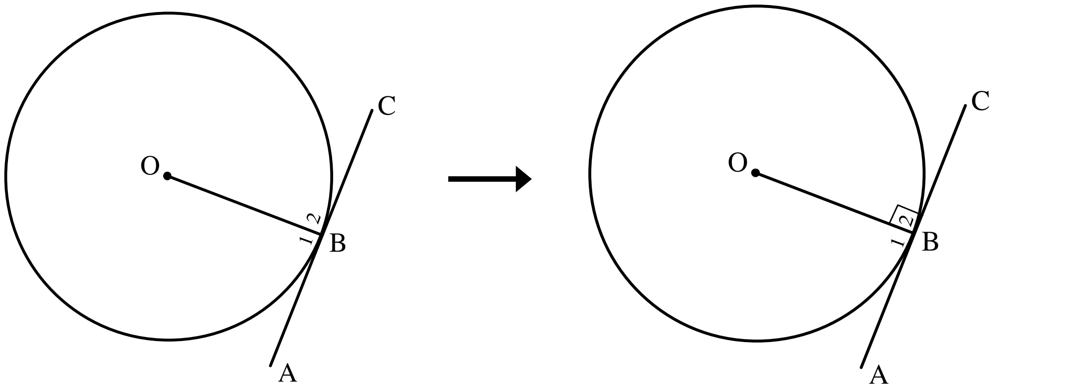
2(b) Write down, with reasons, two cyclic quadrilaterals in the figure.



Statement	Reason

THEOREM 7

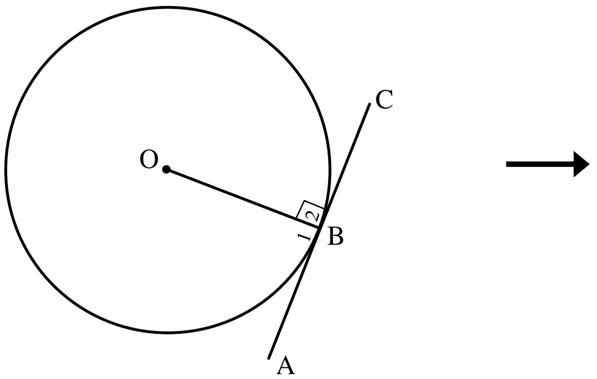
A tangent to a circle is perpendicular to the radius at the point of contact.



If ABC is a tangent to the circle at B, then the radius $OB \perp ABC$, ie. $\hat{B}_1 = \hat{B}_2 = 90°$

CONVERSE OF THEOREM 7

If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then the line is a tangent to the circle.

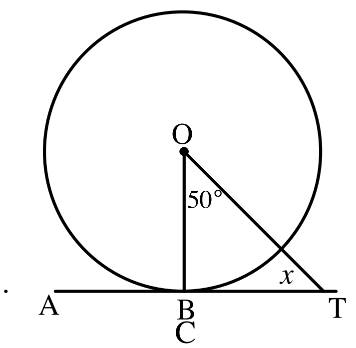


ABC is a tangent

EXAMPLE 9

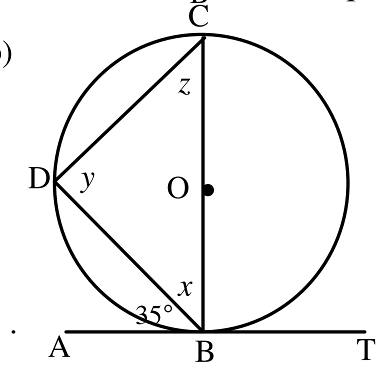
Calculate the value of the unknown angles. O is the centre of each circle and ABT is a tangent.

(a)



Statement	Reason
$O\hat{B}T = 90°$ $\therefore x = 40°$	Tan $\perp$ radius Sum of the angles of Δ

(b)

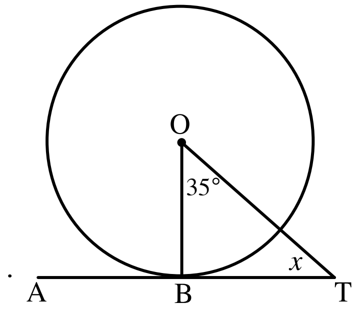


Statement	Reason
$A\hat{B}C = 90°$ $\therefore x = 55°$ $y = 90°$ $z = 35°$	Tan $\perp$ radius $A\hat{B}D = 35°$ $\angle$ in a semi-circle Sum of the angles of Δ

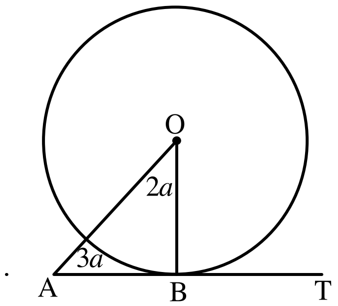
EXERCISE 8

Calculate the value of the unknown angles. O is the centre of each circle and ABT is a tangent.

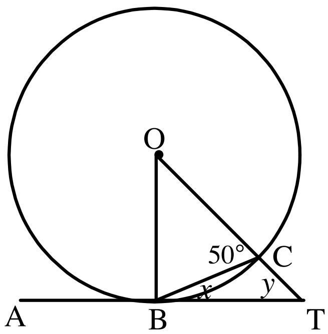
(a)



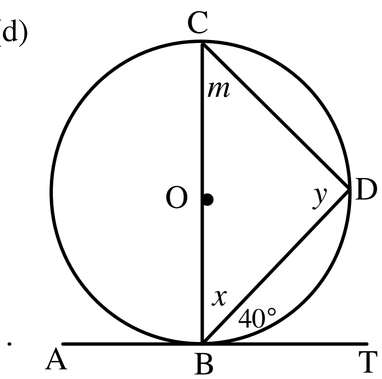
(b)



(c)

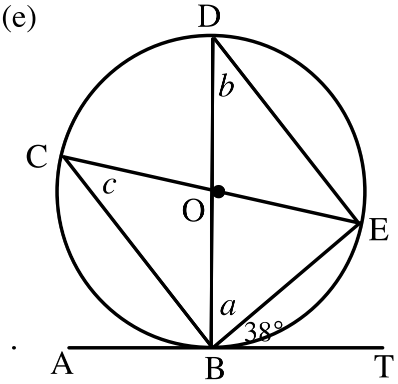


(d)

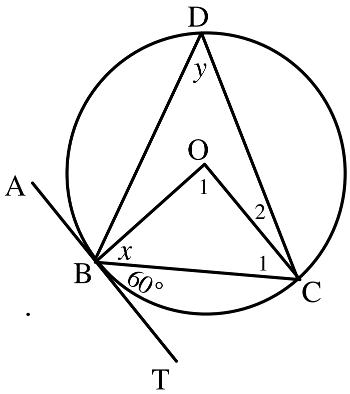


Statement	Reason

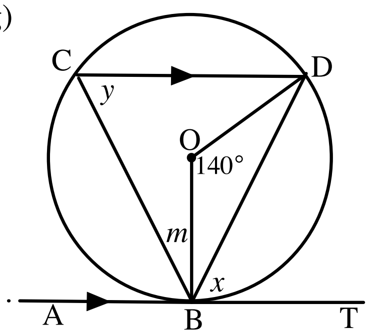
(e)



(f)



(g)



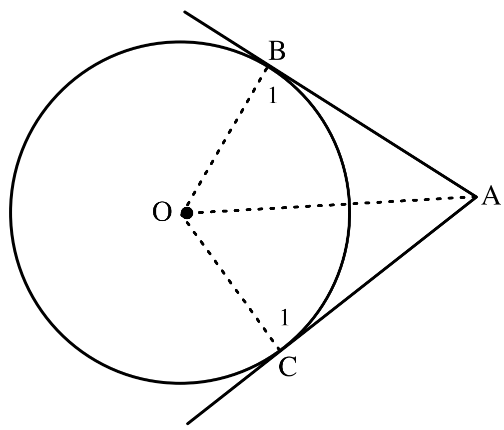
Statement	Reason

<u>THEOREM 8</u>

If two tangents are drawn from the same point outside a circle, then they are equal in length.

Given: Circle with two tangents AB and AC

Required to prove: AB = AC



<u>Proof:</u>

Join OB, OC and OA.

In ΔOBA and ΔOCA:

a) OB = OC ….. radii

b) $\hat{B}_1 = \hat{C}_1 = 90°$ ….. tan ⊥ rad

c) OA = OA ….. common

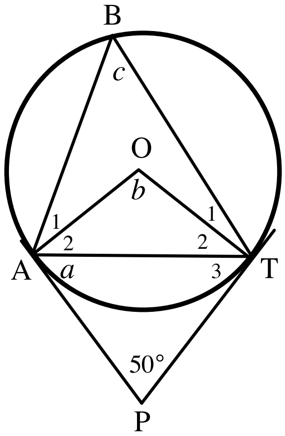
$\therefore \Delta OBA \equiv \Delta OCA$ ….. RHS

$\therefore AB = AC$

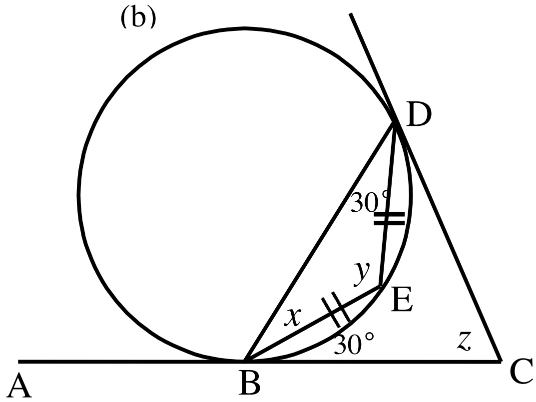
EXERCISE 9

Calculate the value of the unknowns. O is the centre of each circle.

(a)



(b)

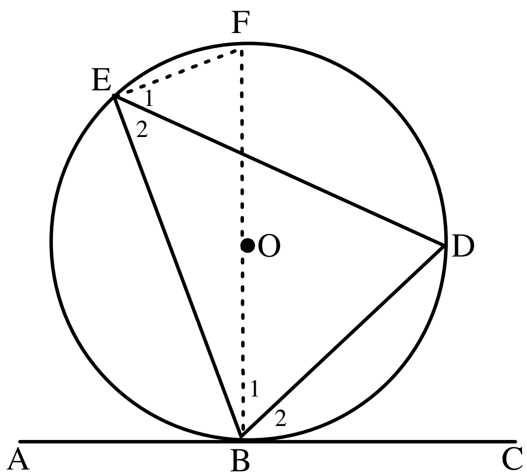


Statement	**Reason**

THEOREM 9

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment. (Tan-chord)

Acute-angle case



Given: Tangent ABC

Required to prove: $C\hat{B}D = B\hat{E}D$

Proof:

Draw diameter BOF and join EF

$\hat{B}_1 + \hat{B}_2 = 90°$ ….. tan ⊥ rad

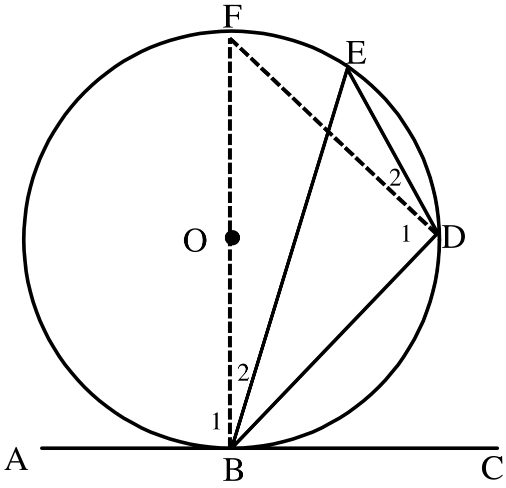
$\hat{E}_1 + \hat{E}_2 = 90°$ ….. ∠ in semi-circle

But $\hat{B}_1 = \hat{E}_1$ ….. FD subt $= \angle s$

$\therefore \hat{B}_2 = \hat{E}_2$

$\therefore C\hat{B}D = B\hat{E}D$

Obtuse-angle case



Given: Tangent ABC

Required to prove: $A\hat{B}E = B\hat{D}E$

Proof:

Draw diameter BOF and join FD

$\hat{B}_1 = 90°$ ….. tan ⊥ rad

$\hat{D}_1 = 90°$ ….. ∠ in semi-circle

$\hat{B}_2 = \hat{D}_2$ ….. FE subt = ∠s

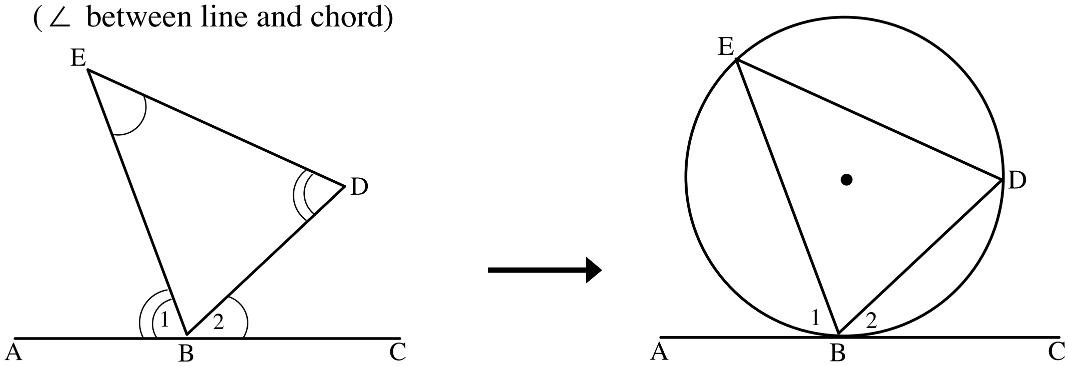
Now $\hat{B}_1 = \hat{D}_1 = 90°$ and $\hat{B}_2 = \hat{D}_2$

$\therefore \hat{B}_1 + \hat{B}_2 = \hat{D}_1 + \hat{D}_2$

$A\hat{B}E = B\hat{D}E$

CONVERSE OF THEOREM 9

If a line is drawn through the endpoint of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.
(∠ between line and chord)



Given: Circle centre O. $D\hat{B}C = B\hat{E}D$

Required to prove: ABC is a tangent

Proof

Draw diameter BOF and join FE

$\hat{B}_2 = \hat{E}_2$ ….. given

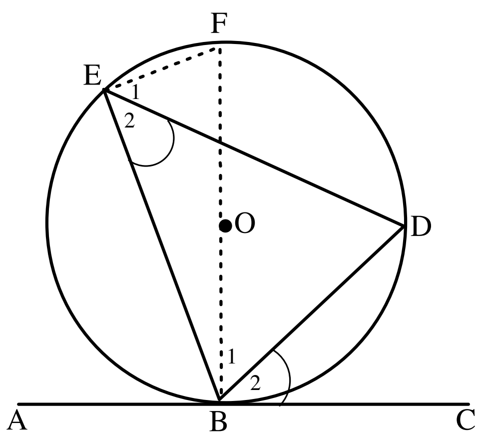
$\hat{B}_1 = \hat{E}_1$ ….. FD subt = ∠s

$\therefore \hat{B}_1 + \hat{B}_2 = \hat{E}_1 + \hat{E}_2$

But $\hat{E}_1 + \hat{E}_2 = 90°$ ….. ∠ in semi-circle

$\therefore \hat{B}_1 + \hat{B}_2 = 90°$

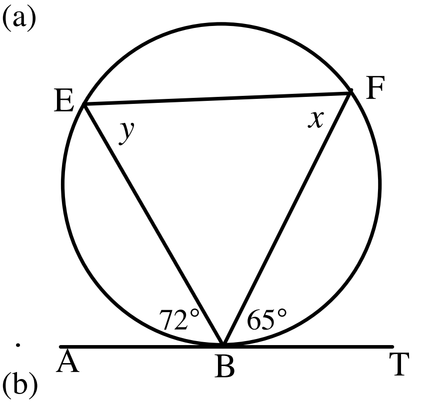
∴ ABC is a tangent ….. rad ⊥ line



EXAMPLE 10

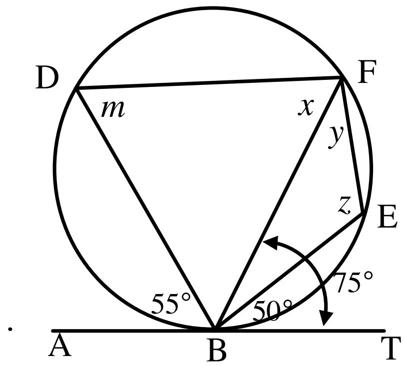
Calculate the value of the unknown angles. ABT is a tangent to the circle.

(a)



Statement	Reason
$x = 72°$ $y = 65°$	Tan chord Tan chord

(b)

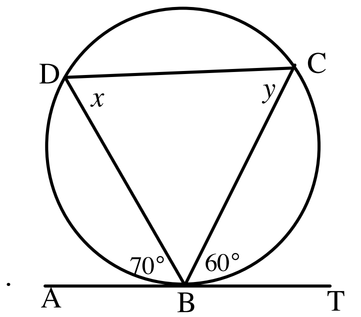


Statement	Reason
$x = 55°$ $y = 50°$ $m = 75°$ $z = 105°$	Tan chord Tan chord Tan chord Opp $\angle s$ of cyclic quad

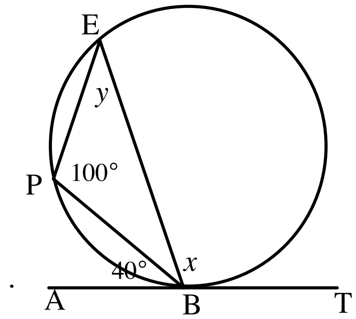
EXERCISE 10

1. Calculate the value of the unknown angles. O is the centre and ABT is a tangent to the circle.

(a)

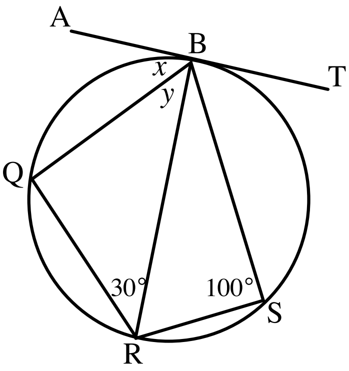


(b)

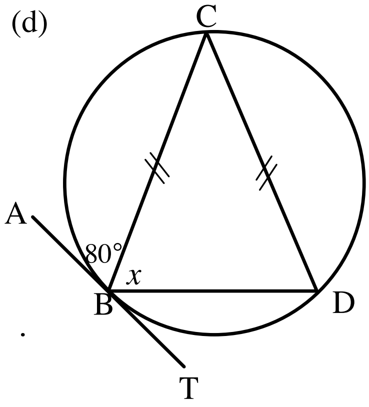


Statement	Reason

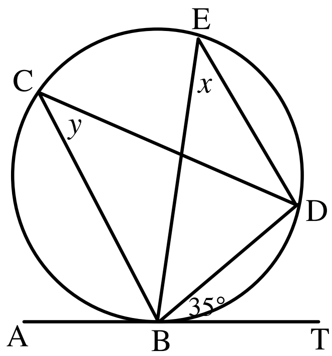
(c)



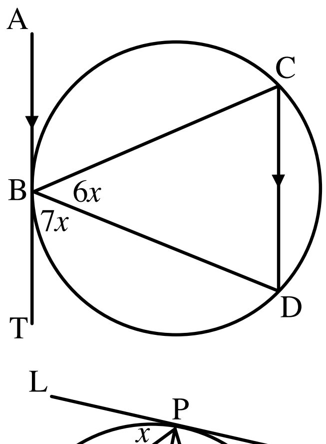
(d)



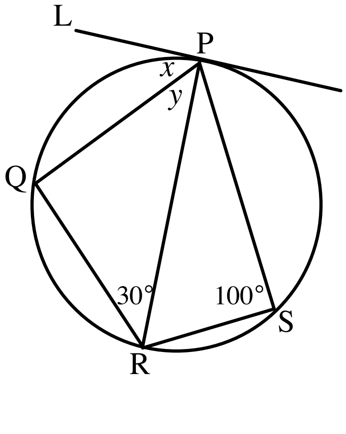
(e)



(f)

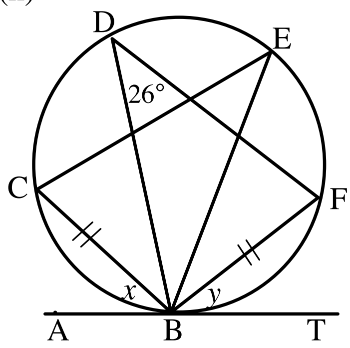


(g)

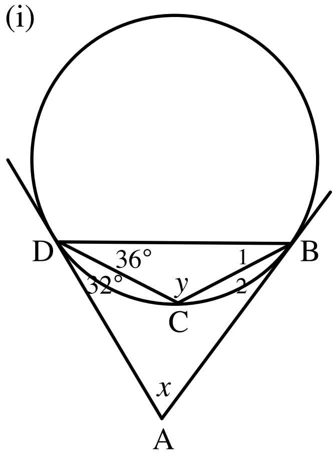


Statement	Reason

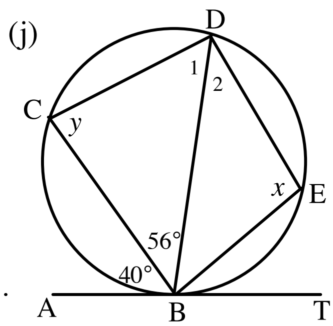
(h)



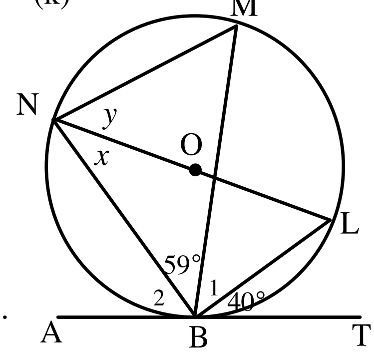
(i)



(j)

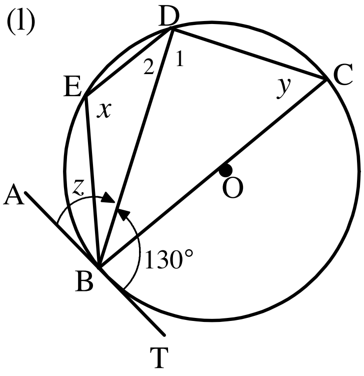


(k)

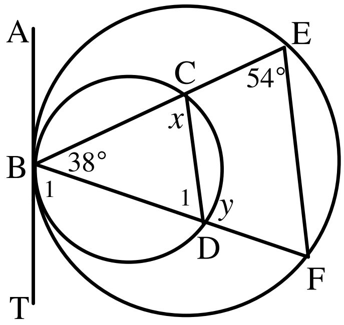


Statement	Reason

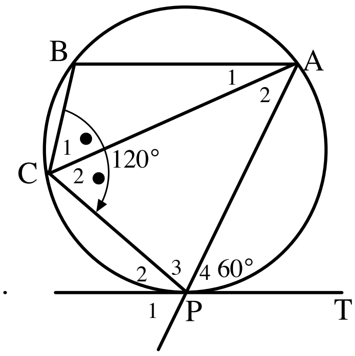
(l)



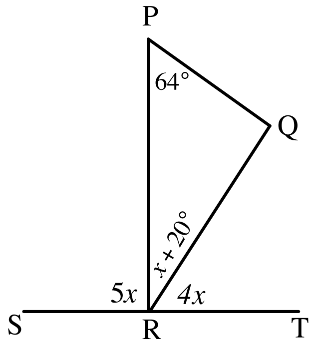
(m)



2(a) Prove that PT is a tangent to the circle if $\hat{P}_4 = 60°$, $B\hat{C}P = 120°$ and $\hat{C}_1 = \hat{C}_2$



2(b) Prove that SRT is a tangent to the circle through PQR.



Statement	Reason

SOLVING GEOMETRICAL PROBLEMS

The following exercises involve the use of all the theorems established thus far.

The following strategy can be used when solving riders:

STEP 1: Analyse the RTP (required to prove) in terms of **angles.**

STEP 2: Pay attention to the **keywords** given. Look for information in the diagram which might prove useful. **Use colours** to mark off equal angles / sides.

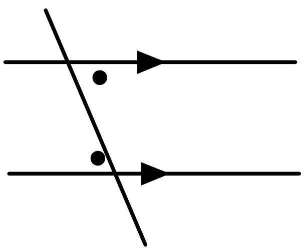
STEP 3: Brainstorm and develop a **rough proof.**

STEP 4: Rewrite a **formal proof.**

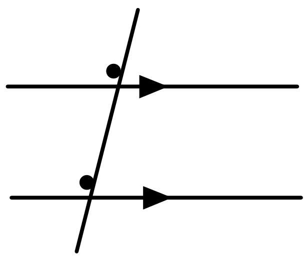
Information and Keyword Mind Map **(For Riders and Numerical exercises)**

1. When parallel lines are given
2. How to prove that lines are parallel
3. Angle or line bisectors
4. Triangle information
5. When you must prove two sides are equal
6. Centre of a circle given
7. Diameter given
8. Angles in the same segment given (NB: angles formed at the circumference)
9. Chords in a circle
10. Cyclic quadrilateral given
11. How to prove that a quadrilateral is cyclic
12. Tangents to circles given
13. How to prove that a line is a tangent to a circle

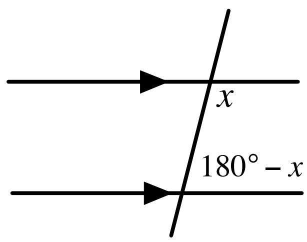
1. When parallel lines are given



alt $\angle s$ =

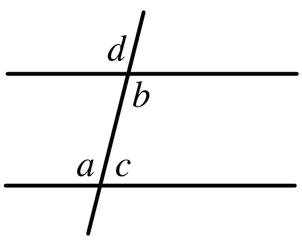


corr $\angle s$ =



co-int $\angle s$ suppl

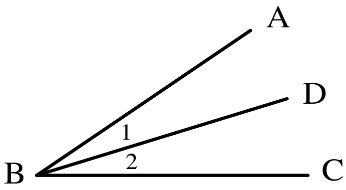
2. How to prove that lines are parallel



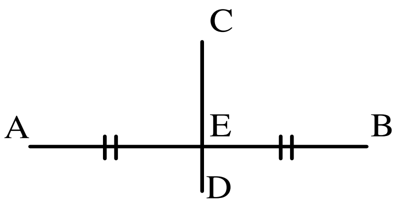
Prove that $a = b$ or $a = d$ or $b + c = 180°$

3. Angle or line bisectors

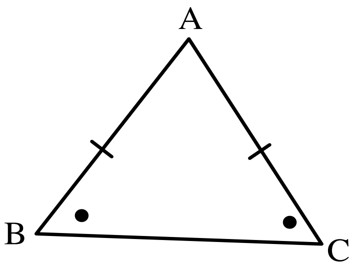
If BD bisects $A\hat{B}C$ then $\hat{B}_1 = \hat{B}_2$



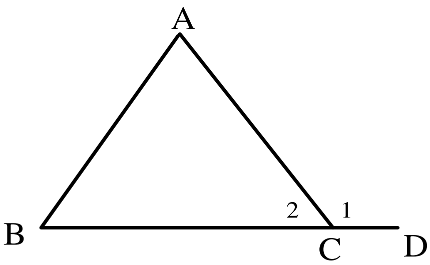
If CD bisects AB then AE = EB



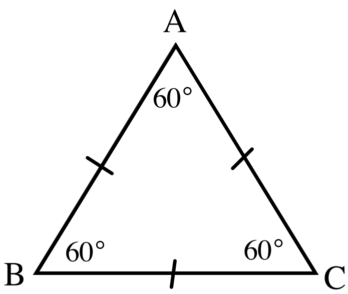
4. Triangle information



If $\hat{B} = \hat{C}$, then AB = AC.
If AB = AC, then $\hat{B} = \hat{C}$.
ΔABC is **isosceles**

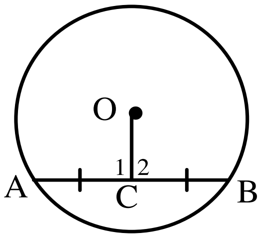


$\hat{A} + \hat{B} + \hat{C}_2 = 180°$ (sum $\angle s$ of Δ)
$\hat{C}_1 = \hat{A} + \hat{B}$ (Ext $\angle$ of Δ)

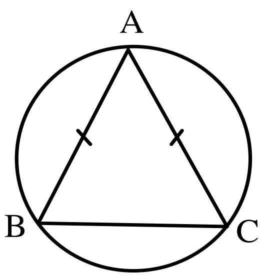


If AB = AC = BC, then $\hat{A} = \hat{B} = \hat{C} = 60°$
ΔABC is **equilateral**

5. When you must prove two sides are equal



or



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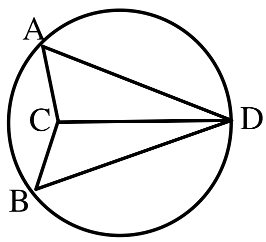
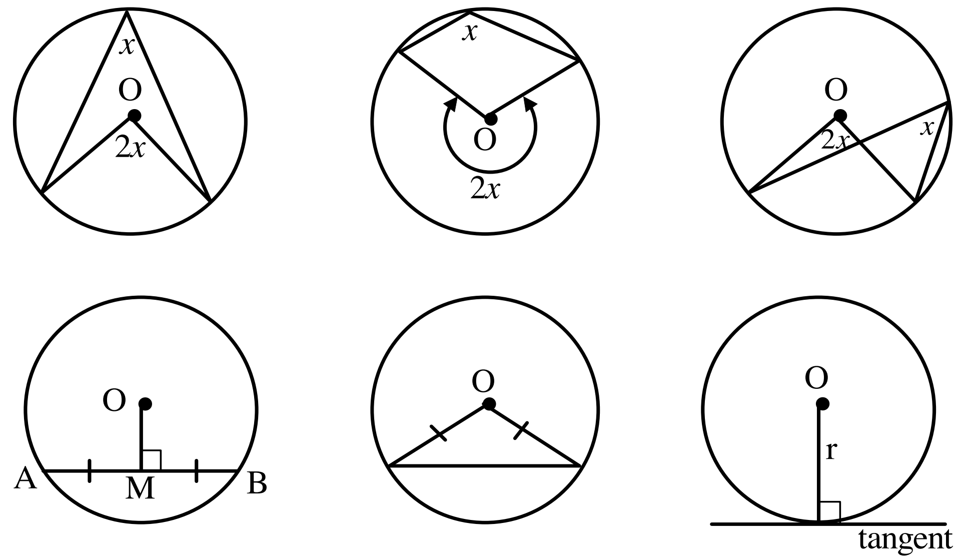


To prove AC = CB, prove $\hat{C}_1 = 90°$

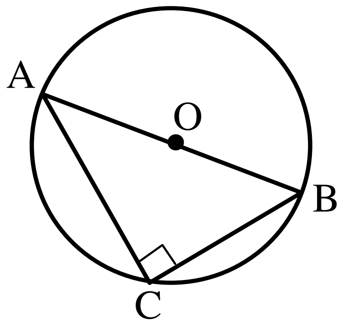
To prove AB = AC, prove $\hat{B} = \hat{C}$

To prove AD = BD, try prove $\Delta ACD \equiv \Delta BCD$

6. Centre of a circle given

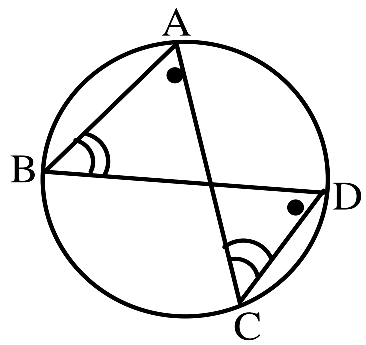


7. Diameter given



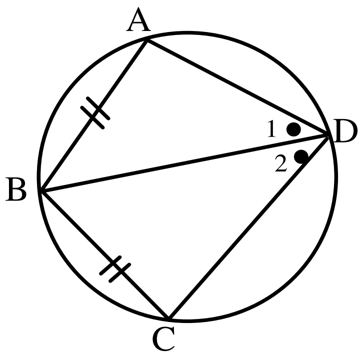
If AOB is diameter then $\hat{C} = 90°$
(∠ in semi circle)

8. Angles in the same segment given (NB: angles formed at the circumference)

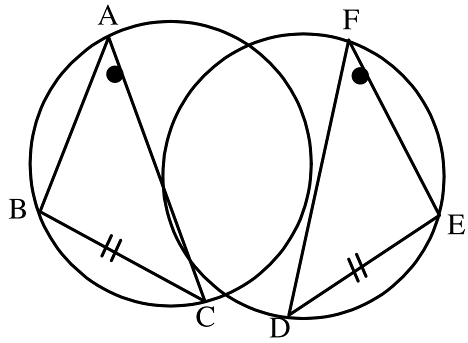


If A, B, C and D are concyclic (lie on a circle) and if AB, AC, BD and CD are chords of the circle, then $\hat{A} = \hat{D}$ and $\hat{B} = \hat{C}$
(∠ in same segment) or
(line/arc subtends equal angles)

9. Chords in a circle

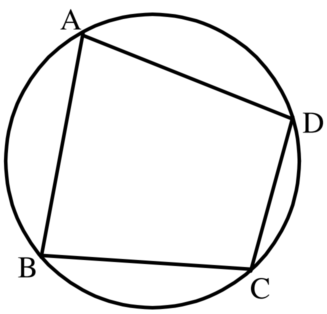


If AB = BC, then $\hat{D}_1 = \hat{D}_2$

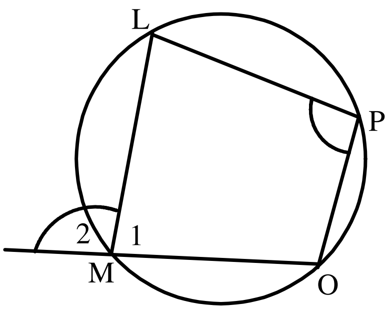


If ABC and DEF are equal circles. then $\hat{A} = \hat{F}$ if BC = DE

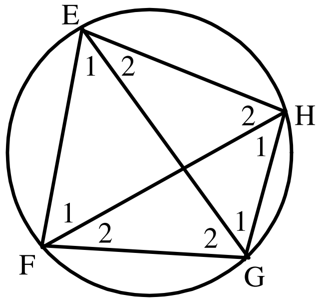
10. Cyclic quadrilateral given



If ABCD is cyclic then
$\hat{A} + \hat{C} = 180°$ and
$\hat{B} + \hat{D} = 180°$
(opp $\angle s$ of cyclic quad)



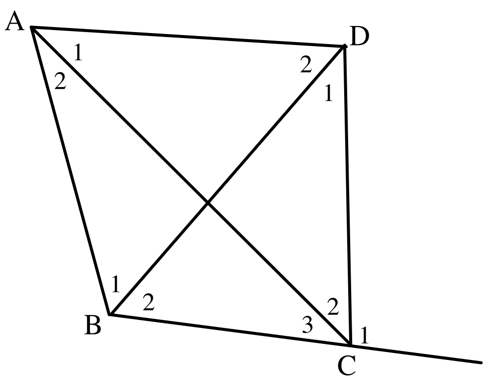
If LMOP is cyclic
then $\hat{M}_2 = \hat{P}$
(Ext $\angle$ cyclic quad)



If EFGH is cyclic then
$\hat{E}_1 = \hat{H}_1$, $\hat{E}_2 = \hat{F}_2$,
$\hat{H}_2 = \hat{G}_2$, $\hat{G}_1 = \hat{F}_1$
($\angle s$ in same segment)

11. How to prove that a quadrilateral is cyclic

ABCD would be a cyclic quadrilateral if you could prove **one** of the following

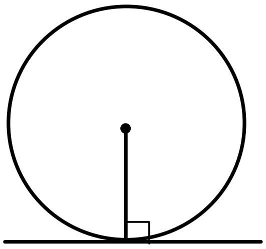


Condition 1:
$\left(\hat{A}_1 + \hat{A}_2\right) + \left(\hat{C}_2 + C_3\right) = 180°$ or
$\left(\hat{B}_1 + \hat{B}_2\right) + \left(\hat{D}_1 + \hat{D}_2\right) = 180°$

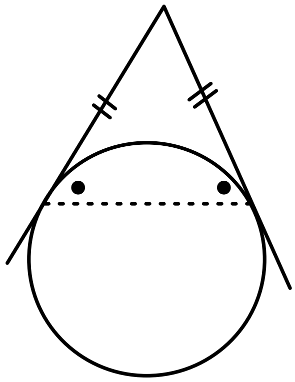
Condition 2:
$\hat{C}_1 = \hat{A}_1 + \hat{A}_2$

Condition 3:
$\hat{A}_1 = \hat{B}_2$ or $\hat{A}_2 = \hat{D}_1$ or $\hat{B}_1 = \hat{C}_2$ or $\hat{D}_2 = \hat{C}_3$

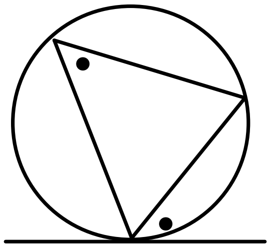
12. Tangents to circles given



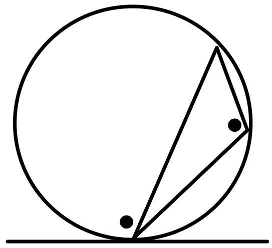
tan ⊥ rad



Tangents from the same point

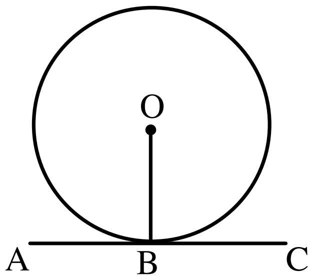


Tan-chord
(acute case)

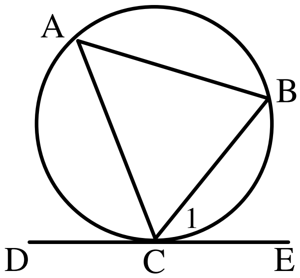


Tan-chord
(obtuse case)

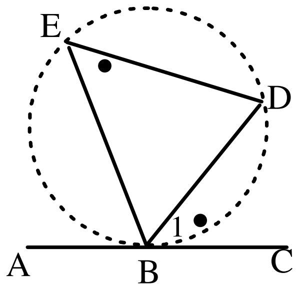
13. How to prove that a line is a tangent to a circle



ABC is a tangent if $O\hat{B}C = 90°$



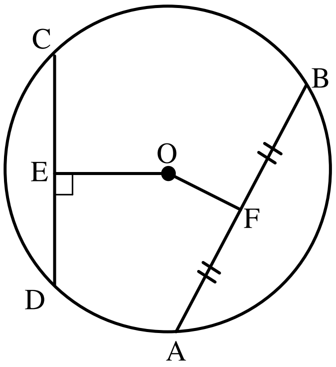
DCE is a tangent if $\hat{C}_1 = \hat{A}$



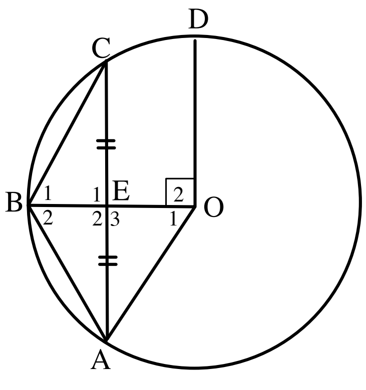
ABC would be a tangent to the “imaginary” circle drawn through EBD if $\hat{B}_1 = \hat{E}$

EXERCISE 11
(MIXED EXERCISE ON NUMERICAL GEOMETRY PROBLEMS)

1. AB and CD are two chords of the circle centre O. $OE \perp CD$, $AF = FB$. $OE = 4\text{cm}$, $OF = 3\text{cm}$ and $AB = 8\text{cm}$. Calculate the length of CD.

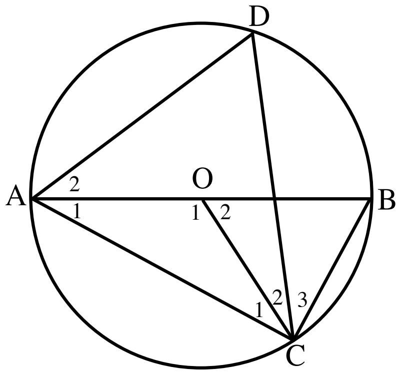


2. In the circle centre O, $BO \perp OD$, $AE = EC$ and $A\hat{O}D = 116°$. Calculate the size of :
   (a) $\hat{C}$
   (b) $A\hat{B}C$

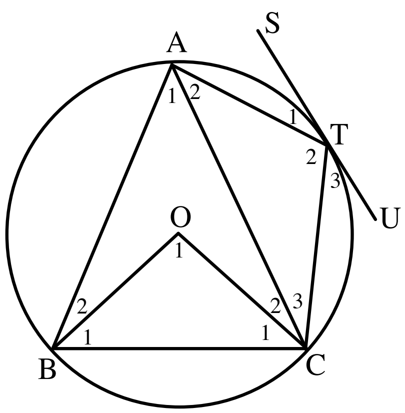


Statement	**Reason**

3. O is the centre of the circle.
AB is the diameter.
$A\hat{O}C = 104°$
and $D\hat{A}B = 32°$.
Calculate the size of:
(a) $\hat{D}$
(b) $\hat{C}_3$
(c) $\hat{A}_1$
(d) $\hat{C}_2$



4. O is the centre of the circle.
STU is a tangent at T.
Chord BC = chord CT,
$A\hat{T}C = 105°$ and
$C\hat{T}U = 40°$.
Calculate the size of :
(a) $\hat{A}_2$
(b) $\hat{A}_1$
(c) $\hat{B}_1 + \hat{B}_2$
(d) $\hat{C}_2$



Statement	**Reason**

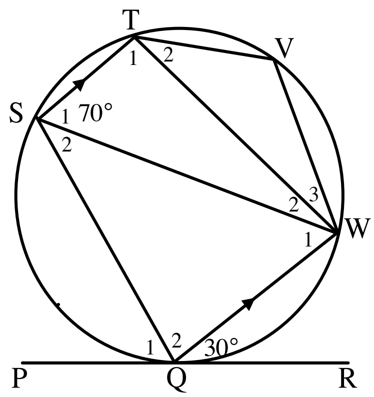
5. PQR is a tangent at Q.
ST || QW.
$W\hat{Q}R = 30°$ and
$T\hat{S}W = 70°$.
Calculate the size of the following angles with reasons:

(a) $\hat{V}$

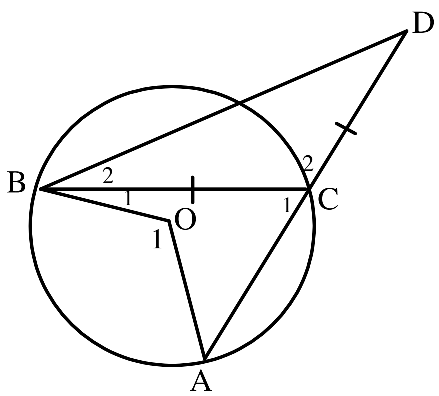
(b) $\hat{Q}_1$

(c) $\hat{T}_1$

(d) $\hat{W}_2$

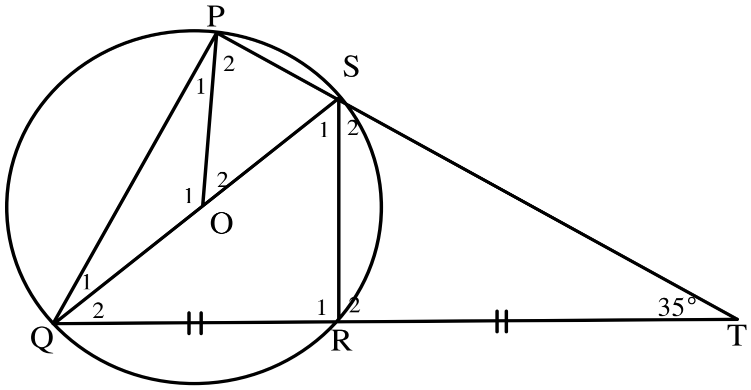


6. O is the centre of circle ABC.
AC is produced to D so that CD = CB.
Prove that $\hat{O}_1 = 128°$ if $\hat{D} = 32°$.



Statement	Reason

7. QOS is a diameter of the circle centre O. QR = RT and $\hat{T} = 35°$.
Calculate, with reasons, the size of:
(a) $\hat{R}_2$ (b) $\hat{S}_2$ (c) $\hat{Q}_2$ (d) $\hat{Q}_1$ (e) $\hat{P}_1$

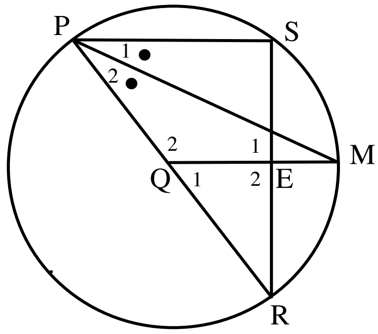


Statement	Reason

EXAMPLES ON SOLVING RIDERS

The following examples cover the most common techniques used in geometry. By studying, mastering and applying the techniques, you will be able to achieve a great deal of success in solving riders.

EXAMPLE 11



PR is a diameter of circle PRMS with centre Q. PS, SR and PM are chords. PM bisects $R\hat{P}S$. Prove that:

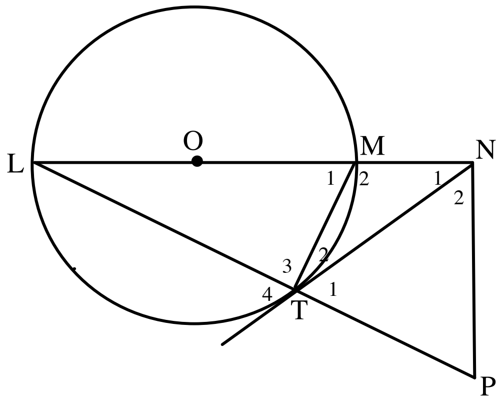
(a) PS ∥ QM

(b) QM ⊥ SR

(c) QM bisects SR

Statement	Reason
(a) $\hat{P}_2 = \hat{M}$	∠ s opp equal radii
$\hat{P}_2 = \hat{P}_1$	given
$\therefore \hat{P}_1 = \hat{M}$	
∴ PS∥QM	alternate angles equal
(b) $\hat{S} = 90°$	∠ in semi-circle
$\therefore \hat{E}_2 = 90°$	Corresponding angles equal
∴ QM ⊥ SR	
(c) RE = ES	Perpendicular from centre to chord
∴ QM bisects SR	

EXAMPLE 12



LOM is a diameter of circle LMT. The centre is O. TN is a tangent at T. LN ⊥ NP.
MT is a chord. LT is a chord produced to P.
Prove that:

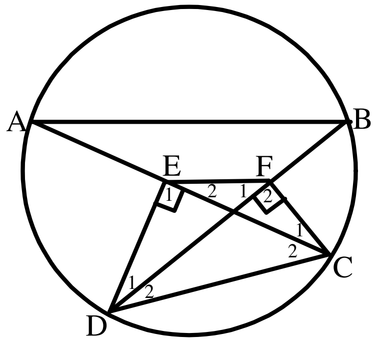
(a) MNPT is a cyclic quadrilateral

(b) NP = NT

Statement	Reason
(a) $\hat{N}_1 + \hat{N}_2 = 90°$	Given
$\hat{T}_3 = 90°$	∠ in semi-circle
$\therefore \hat{N}_1 + \hat{N}_2 = \hat{T}_3$	
∴ MNPT is a cyclic quad	Ext ∠ = int opp ∠
(b) $\hat{T}_1 = \hat{T}_4$	Vertically opp angles
$\hat{T}_4 = \hat{M}_1$	Tan chord
$\hat{M}_1 = \hat{P}$	Ext ∠ of cyclic quad
$\therefore \hat{T}_1 = \hat{P}$	
∴ NP = NT	Sides opp equal ∠ s

EXAMPLE 13

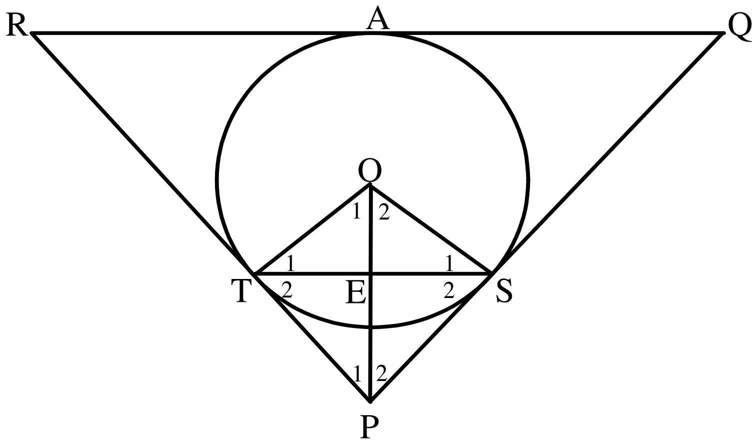
AB, AC, DB, and DC are chords.
DE ⊥ AC and DB ⊥ FC.
Prove that:
(a) DEFC is a cyclic quadrilateral (b) AB ∥ EF



Statement	Reason
(a) $\hat{E}_1 = 90°$	Given
$\hat{F}_2 = 90°$	Given
∴ DEFC is a cyclic quadrilateral	DC subtends = ∠ s
(b) $\hat{E}_2 = \hat{D}_2$	FC subtends = ∠ s
$\hat{A} = \hat{D}_2$	Arc BC subtends = ∠ s
∴ $\hat{E}_2 = \hat{A}$	
∴ AB ∥ EF	Corr ∠ s =

EXAMPLE 14

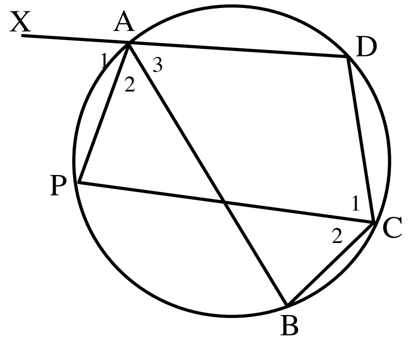
O is the centre of circle SAT which is inscribed in ΔPQR. PQ, QR and PR are tangents to the circle. Prove:
(a) PSOT is a cyclic quadrilateral (b) OS is a tangent to circle SPE



Statement	Reason
(a) $\hat{T}_1 + \hat{T}_2 = 90°$	Tan ⊥ Radius
$\hat{S}_1 + \hat{S}_2 = 90°$	Tan ⊥ Radius
∴ $\hat{T}_1 + \hat{T}_2 + \hat{S}_1 + \hat{S}_2 = 180°$	
∴ PSOT is a cyclic quadrilateral	Opp angles supplementary
(b) $\hat{S}_1 = \hat{T}_1$	∠ s opp equal radii
$\hat{P}_2 = \hat{T}_1$	OS subtends = ∠ s
∴ $\hat{S}_1 = \hat{P}_2$	
∴ OS is a tangent to circle SPE	∠ between line and chord

EXAMPLE 15

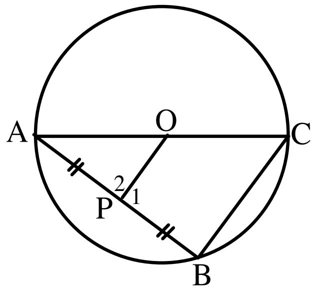
A, D, C, B and P are concyclic. PC bisects $D\hat{C}B$.
Prove that PA bisects $X\hat{A}B$



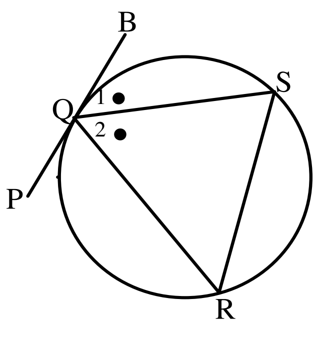
Statement	Reason
$\hat{A}_1 = \hat{C}_1$	Ext ∠ of cyclic quad
$\hat{C}_1 = \hat{C}_2$	Given
$\hat{C}_2 = \hat{A}_2$	PB subtends = ∠ s
$\therefore \hat{A}_1 = \hat{A}_2$	

EXERCISE 12 **(SOLVING RIDERS)**

1. AC is a diameter of circle centre O.
   B is a point on the circle.
   OP bisects AB.
   Prove that OP ∥ BC.

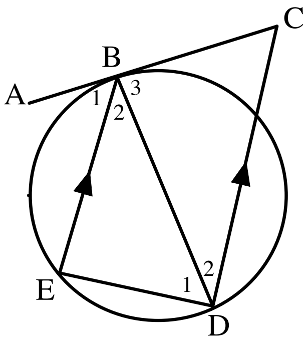


2. PQB is a tangent to the circle QRS at Q.
   QS bisects $B\hat{Q}R$.
   Prove that QS = RS

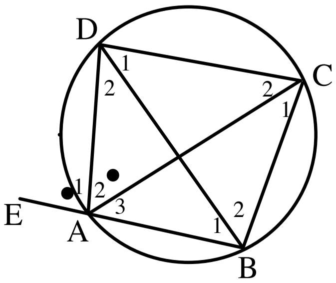


Statement	Reason

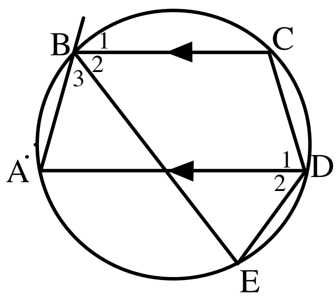
3. ABC is a tangent to the circle BED. BE ∥ CD.
Prove that $\hat{D}_1 = \hat{C}$



4. ABCD is a cyclic quadrilateral.
BA is produced to E.
AD bisects $E\hat{A}C$.
Prove that DC = DB.

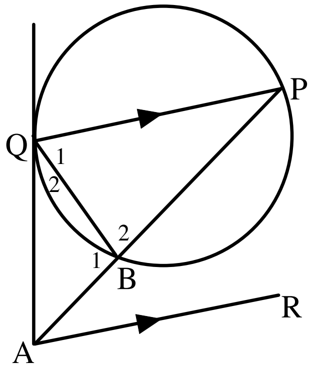


5. In circle ABCDE, BC and AD are parallel chords.
AB is produced to F.
(a) Name two cyclic quadrilaterals
(b) Prove that:
(1) $\hat{B}_1 = \hat{E}$
(2) $\hat{D}_1 = \hat{A}$

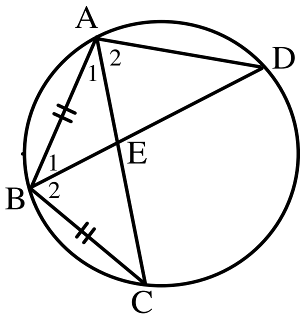


Statement	**Reason**

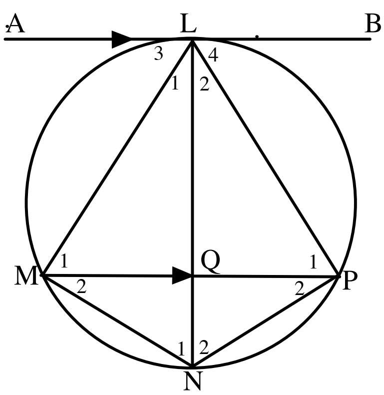
6. AQ is a tangent to the circle in Q and QP || AR. Prove that RA is a tangent to circle ABQ at A.



7. In circle ABCD, AB = BC. Prove that AB is a tangent to circle AED in A.

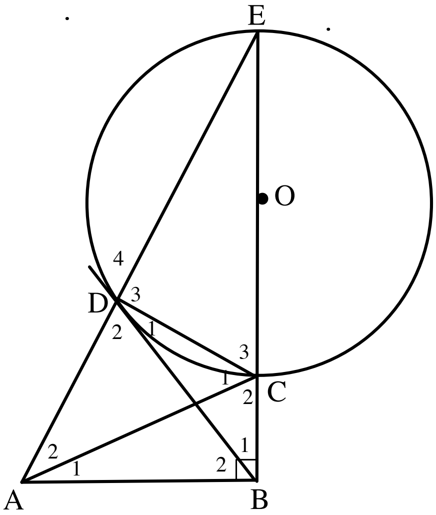


8. ALB is a tangent to circle LMNP. ALB || MP.
Prove that:
(a) LM = LP
(b) LN bisects $M\hat{N}P$
(c) LM is a tangent to circle MNQ

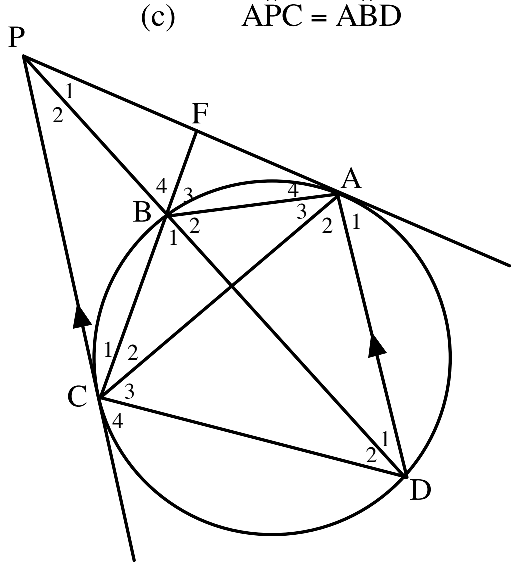


Statement	Reason

9. EC is a diameter of circle DEC.
EC is produced to B.
BD is a tangent at D.
ED is produced to A and $AB \perp BE$.
Prove that:
(a) ABCD is a cyclic quadrilateral
(b) $\hat{A}_1 = \hat{E}$
(c) ΔBDA is isosceles
(d) $\hat{C}_2 = \hat{C}_3$

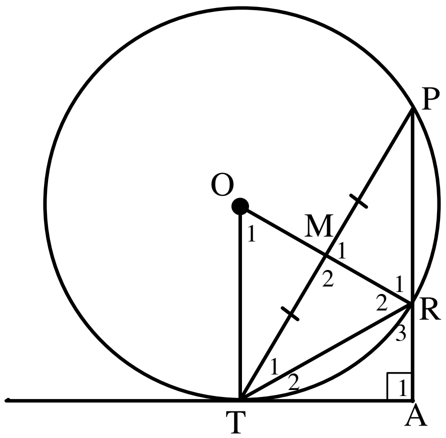


10. PA and PC are tangents to the circle at A and C. AD ∥ PC, and PD cuts the circle B. CB is produced to meet AP at F. AB, AC and DC are drawn.
Prove that:
(a) AC bisects $P\hat{A}D$
(b) $\hat{B}_1 = \hat{B}_3$
(c) $A\hat{P}C = A\hat{B}D$

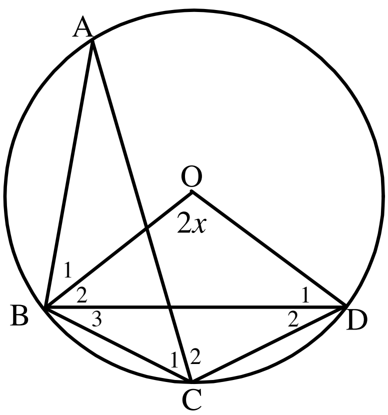


Statement	Reason

11. TA is a tangent to the circle PRT. M is the midpoint of chord PT. The centre of the circle is O. PR is produced to intersect TA at A and $TA \perp PA$. T and R are joined. OR and OT are radii.
Prove that:
(a) MTAR is a cyclic quadrilateral
(b) $PR = RT$
(c) TR bisects $P\hat{T}A$
(d) $T_2 = \frac{1}{2}\hat{O}_1$



12. O is the centre of the circle through A, B, C and D. $BC = CD$ and $B\hat{O}D = 2x$. Express the following in terms of x.
(a) $\hat{B}_2$
(b) $B\hat{C}D$
(c) $\hat{A}$

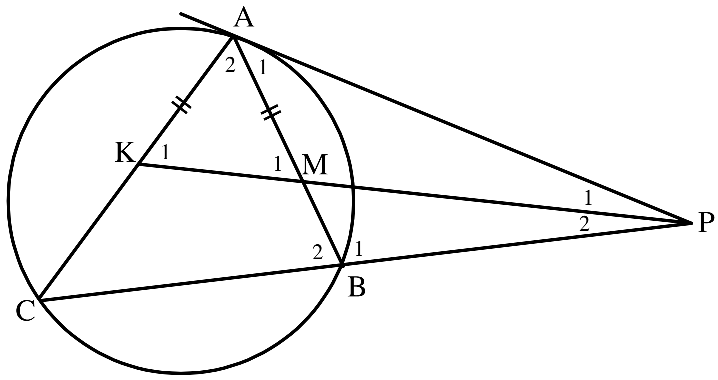


Statement	Reason

MORE ADVANCED RIDERS

EXAMPLE 16

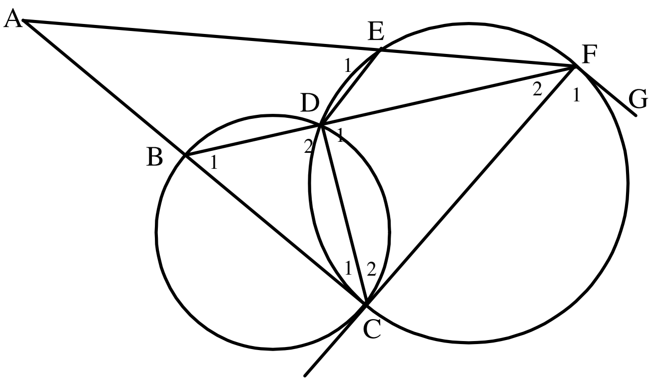
In the diagram, PA is a tangent to the circle at A. AC and AB are chords and PM is produced to K such that AK = AM . K and M lie on AC and AB respectively. Chord CB is produced to P.
Prove that KP bisects $A\hat{P}C$



Statement	Reason
$\hat{P}_1 + \hat{A}_1 = \hat{M}_1$	Ext ∠ of ΔAMP
But $\hat{M}_1 = \hat{K}_1$	∠ 's opposite equal sides
$\therefore \hat{P}_1 + \hat{A}_1 = \hat{K}_1$	
But $\hat{P}_2 + \hat{C} = \hat{K}_1$	Ext ∠ of ΔAMP
and $\hat{A}_1 = \hat{C}$	Tan-chord
$\therefore \hat{P}_1 = \hat{P}_2$	

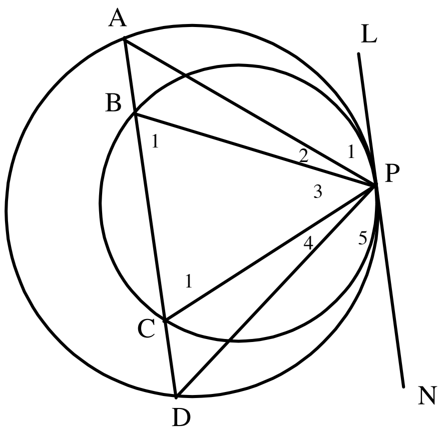
EXERCISE 13

1. FC is a tangent to circle BCD and FG is a tangent to circle FEDC. The circles intersect at D and C. Chord BD is produced to F. DC is joined. Chord FE and chord CB produced meet at A. Prove that:
   (a) ABDE is a cyclic quadrilateral.
   (b) $\hat{D}_1 = \hat{C}_1 + \hat{C}_2$
   (c) FG||AC

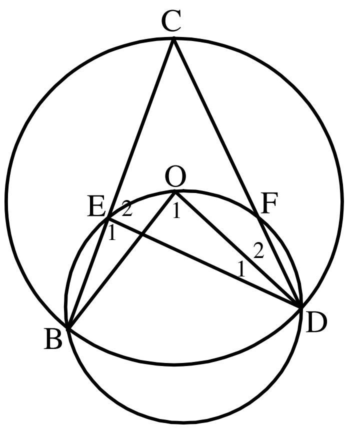


Statement	Reason

2. LPN is a tangent to circle ADP. Circle BCP touches the larger circle internally at P. Chord AD cuts the smaller circle at B and C and BP and CP are joined. Prove that $\hat{P}_2 = \hat{P}_4$



3. O is the centre of the larger circle. The second circle drawn through O intersects the larger circle at B and D. Chords BC and DC intersect the smaller circle at E and F respectively. Prove that DE = EC.



Statement	**Reason**

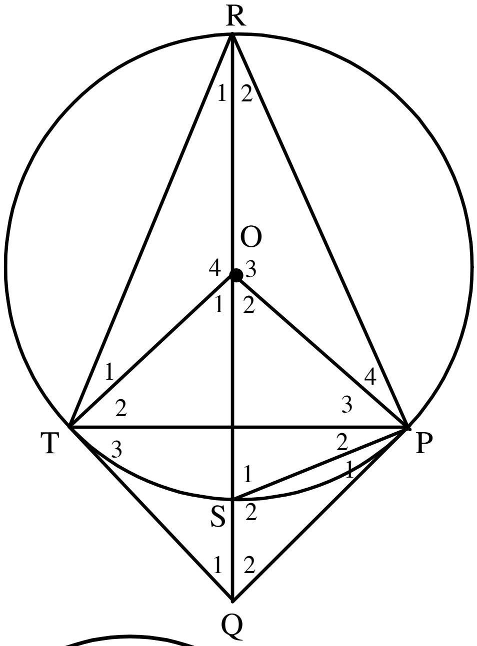
<u>REVISION EXERCISE</u>

1 In the diagram below, QP is a tangent to a circle with centre O. RS is a diameter of the circle and RQ is a straight line. T is a point on the circle. PS bisects $T\hat{P}Q$ and $S\hat{P}Q = 22°$. Calculate the following, giving reasons:

(a) $\hat{P}_2$ (b) $\hat{R}_2$

(c) $\hat{P}_3 + \hat{P}_4$ (d) $\hat{R}_1$

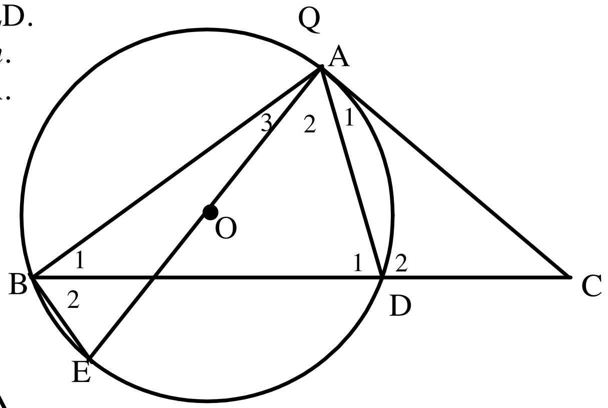
(e) $\hat{O}_1$ (f) $\hat{Q}_2$



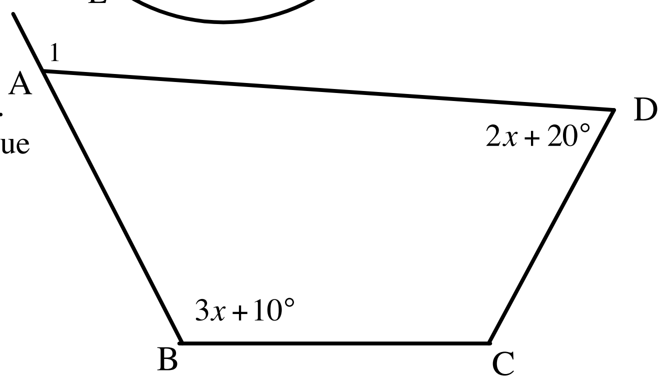
2. O is the centre of the circle ABED. The radius of the circle is 6,5 *cm*. AC is a tangent to the circle at A. BD is produced to C such that AD = DC.

(a) Prove that AB = AC.

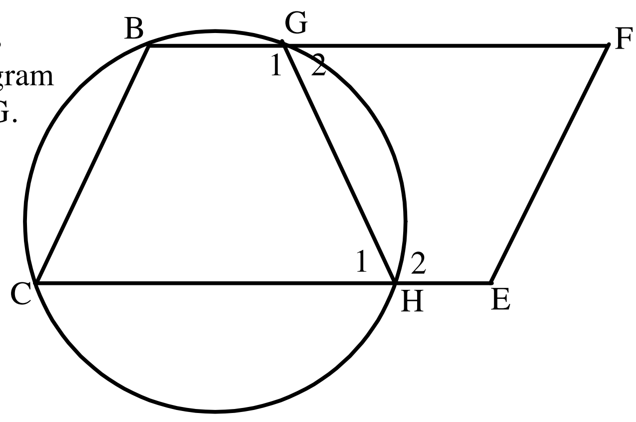
(b) If BE = 5*cm*, calculate the length of AC.



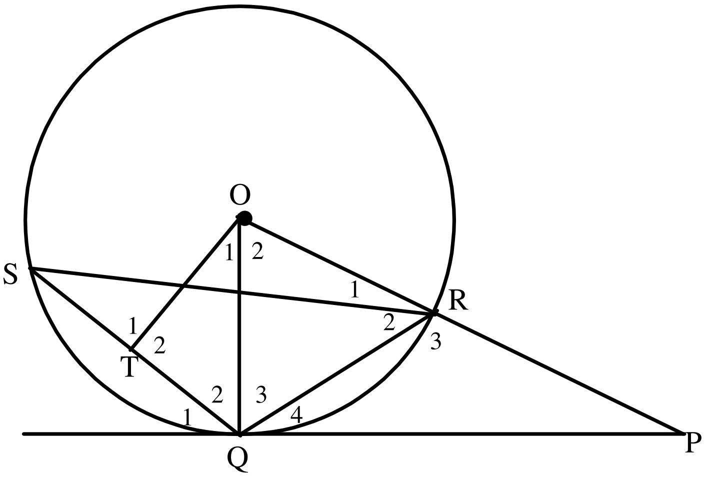
3. In the diagram below, $\hat{A}_1 = \hat{C}$, $\hat{B} = 3x + 10°$ and $\hat{D} = 2x + 20°$. Calculate, with reasons, the value of x.



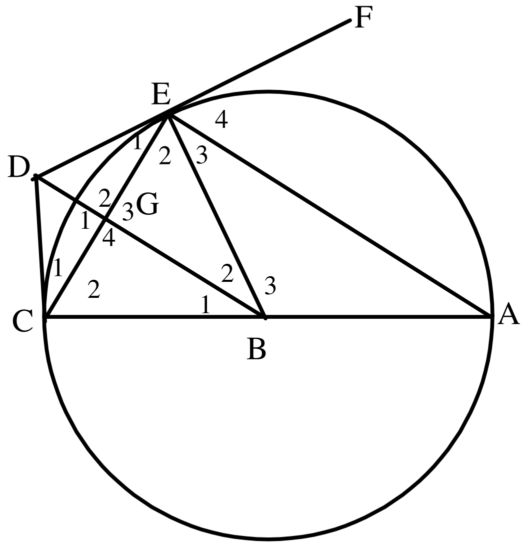
4. A circle through B and C cuts sides BF and CE of parallelogram BCEF respectively at H and G. Prove that FEHG is a cyclic quadrilateral.



5. In the figure, O is the centre of the circle. PRO is a straight line intersecting the circle at R. OT is perpendicular to chord SQ at T. SR is joined. Radius OQ and line PQ meet at Q. $PQ = 12$ units, $QS = 8$ units, $RP = 8$ units and $OT = 3$ units.
   - (a) Prove that PQ is a tangent to the circle at Q.
   - (b) Express $\hat{P}$ in terms of x if $\hat{Q}_4 = x$.

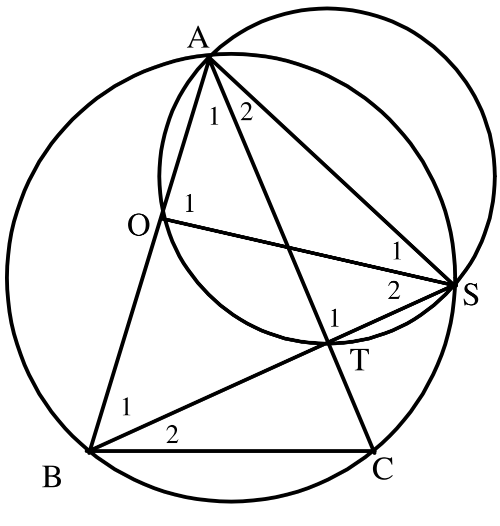


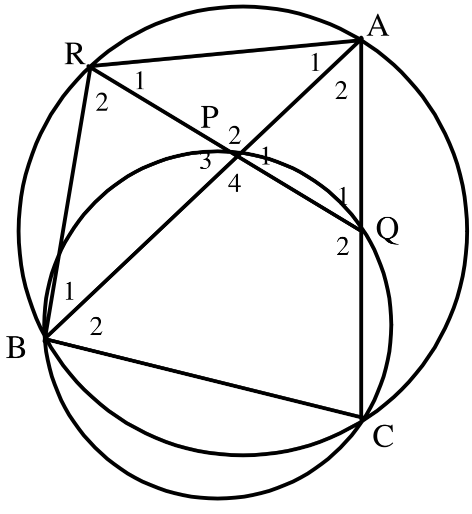
6. AC is a diameter of the circle centre B. FED is a tangent to the circle at E and $BG \perp EC$. BG produced cuts FE produced at D. DC is drawn. Prove that:
   - (a) BG||AE
   - (b) BCDE is a cyclic quadrilateral.
   - (c) DC is a tangent to circle EAC.
   - (d) DC is a tangent to circle BCG.



7. Circles OAS and ABCS intersect at A and S. SB bisects $A\hat{B}C$. Chord AT is produced to C. Prove that $SA = SO$

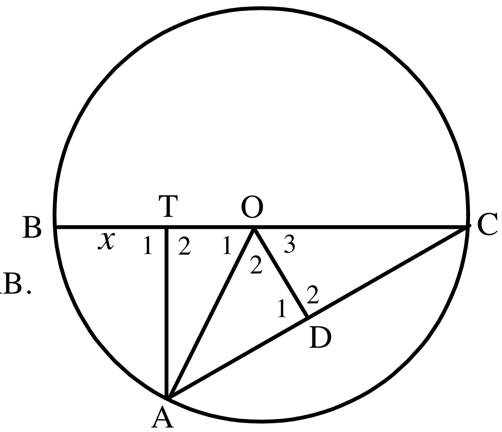
8. P is a point on side AB of ΔABC. The circle through P, B and C cuts AC in Q. QP produced cuts the larger circle at R. Prove that $\hat{P}_1 = \hat{A}_1 + \hat{B}_1$



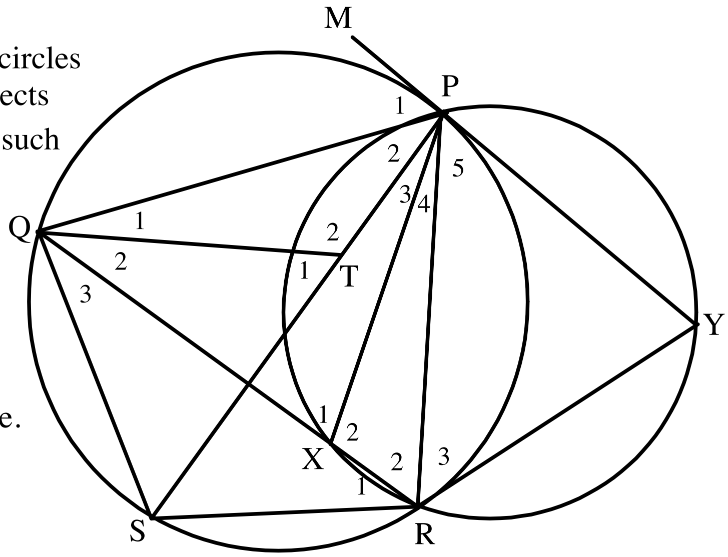


SOME CHALLENGES

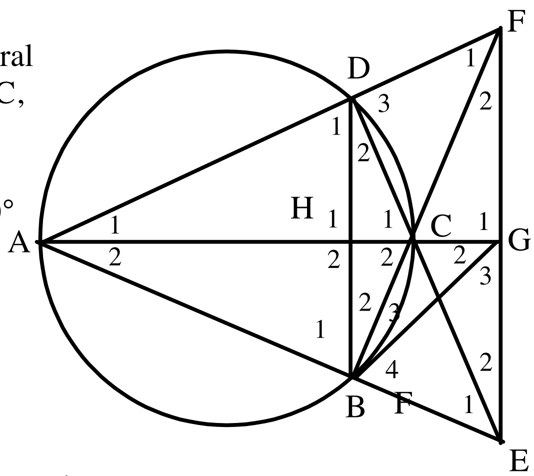
1. BC is a diameter of circle ABC with centre O. OD ⊥ AC and AT ⊥ BOC. BT = x, TO = 5 and AD = $\sqrt{42}$.
   (a) Prove that OA = 7
   (b) Calculate the length of AB.



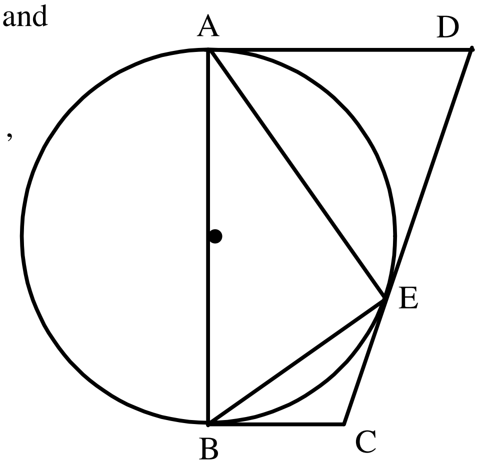
2. PR is a common chord of circles PQSR and PXRY. PR bisects $X\hat{R}Y$. T lies on chord PS such that ST = SR = SQ. MPY touches the larger circle at P. Prove that:
   (a) $\hat{Q}_1 = \hat{Q}_2$
   (b) QP is a tangent to the smaller circle.



3. In the figure, ABCD is a cyclic quadrilateral with AB = AD and DC = BC. DC and BC, both produced meet AB and AD, both produced, at E and F respectively. AC produced meets FE at G with $\hat{G}_1 = 90°$. Prove that:
   (a) AC is a diameter of the circle.
   (b) DBEF is a cyclic quadrilateral.
   (c) BC bisects $D\hat{B}G$.



4. In the figure, AB is a diameter of the circle and DA, CB and DEC are tangents.
   (a) Prove that AD||BC
   (b) If AB = 12 units and CD = 13 units, determine, with reasons, the area of trapezium ABCD.

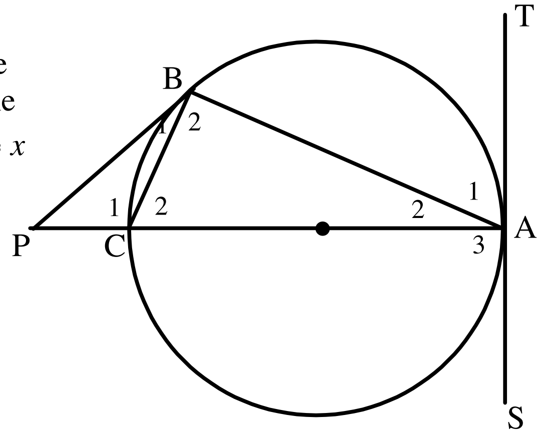


5. In the sketch, AC is a diameter of the circle and PB is a tangent. AC produced meets the tangent at P. TAS is also a tangent and $\hat{P} = x$ and $\hat{C}_1 = y$.

Prove that:

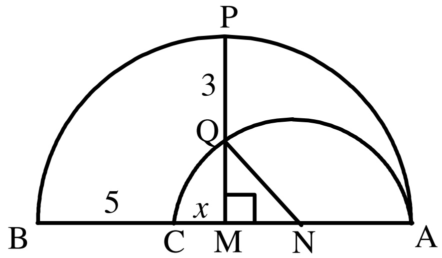
(a) $\hat{A}_2 + \hat{A}_3 = y$

(b) $x + 2y = 270°$



6. In the sketch, M and N are the centres of the two semi-circles. $MP \perp AB$, $BC = 5$ units, $PQ = 3$ units and $MC = x$ units.

Calculate the radius of each semi-circle.

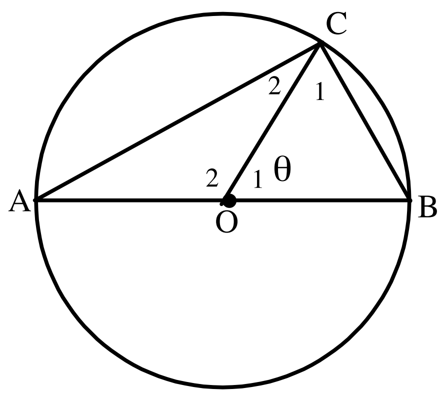


7. In the figure, AB is a diameter of the circle with centre O and radius r. C is a point on the circle such that $C\hat{O}B = \theta$ (acute angle).

Prove that

(a) $BC = \dfrac{r\sin\theta}{\cos\frac{\theta}{2}}$

(b) $AC^2 = 2r^2(1+\cos\theta)$



CHAPTER 9 – PROBABILITY

REVISION OF GRADE 10 CONCEPTS

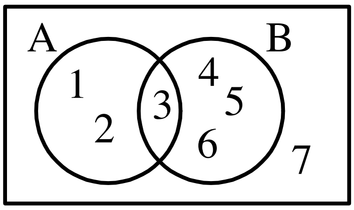
(1) For **inclusive** events A and B, the following rule is true:

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

where $P(A \text{ and } B) \neq 0$

Alternative notation:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Notice that $A = \{1 ; 2 ; 3\}$ and not $A = \{4 ; 5 ; 6 ; 7\}$ (7 is not in A or B)

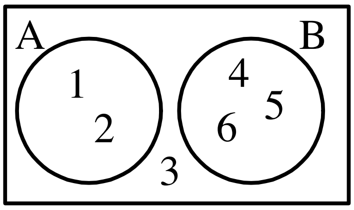
(2) For two **mutually exclusive** events A and B, the following rule is true:

$P(A \text{ or } B) = P(A) + P(B)$

where $P(A \text{ and } B) = 0$

Alternative notation:

$P(A \cup B) = P(A) + P(B)$ where $P(A \cap B) = 0$

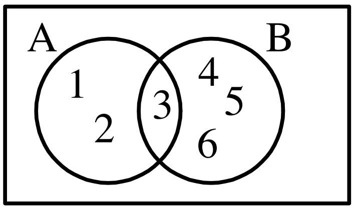


Notice that $A = \{1 ; 2\}$ and not $A = \{3 ; 4 ; 5 ; 6\}$ (3 is not in A or B)

(3) Events A and B are **exhaustive** if

$P(A \text{ or } B) = 1$

Inclusive as well as mutually exclusive events can be exhaustive if together they contain all elements of the sample space.

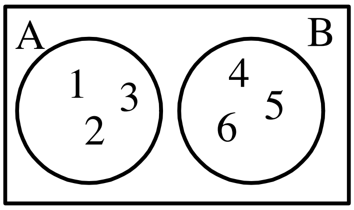


Notice that $A = \{1 ; 2 ; 3\}$ and not $A = \{4 ; 5 ; 6\}$

(All elements that are not in A are in B)

(4) Mutually exclusive, exhaustive events are called **complementary events.**

The statement "not A" (or A^{I}) is called the complement of A.



For **<u>any</u>** two **complementary events**:

$P(\text{not } A) + P(A) = 1$ or $P(\text{not } A) = 1 - P(A)$

Alternative notation: $P(A) + P(A^{I}) = 1$ or $P(A^{I}) = 1 - P(A)$

In example (4), $P(A) + P(B) = \frac{3}{6} + \frac{3}{6} = 1$ (A and B are complementary)

In example (2) above, $P(A) + P(B) = \frac{2}{6} + \frac{3}{6} \neq 1$ (A and B are not complementary since B doesn't contain all elements not in A).

In example (3), $P(A) + P(B) = \frac{3}{6} + \frac{4}{6} \neq 1$ (A and B are not complementary)

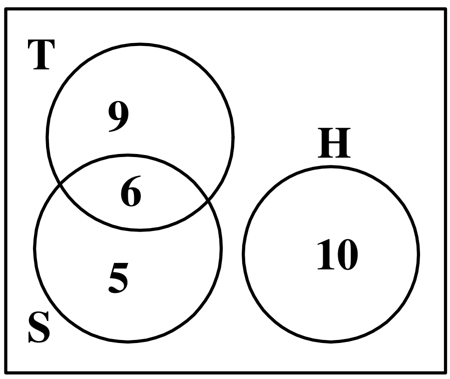
EXERCISE 1 (REVISION)

1. In a recent survey, it was found that 90 people supported Kaiser Chiefs only, 80 people supported Orlando Pirates only and 5 people supported both teams. There were 10 people who did not support either team.
   (a) Draw a venn diagram to illustrate this information.
   (b) Determine the probability that a person selected at random will support Kaiser Chiefs only.
   (c) Determine the probability that a person selected at random will support both teams.
   (d) Determine the probability that a person selected at random will support none of the teams.
   (e) Determine whether the events involved are inclusive or mutually exclusive?
   (f) Are these events complementary? Give a reason.

2. In the recent municipal elections in a certain town, there were 5014 votes for Candidate A, 3702 for Candidate B and 1215 for Candidate C.
   (a) Draw a venn diagram to illustrate this information.
   (b) Determine the probability that a voter selected at random voted for Candidate A.
   (c) Determine whether the events in this situation are inclusive or mutually exclusive. Give reasons.
   (d) Are these events complementary? Give a reason.

3. In a survey conducted by a local franchise selling pies, it was found that of the 220 customers, 170 bought chicken pies, 70 bought meat pies and 20 bought both.
   (a) Draw a venn diagram to illustrate this information.
   (b) Determine the probability that a customer bought a chicken pie only?
   (c) Determine the probability that a customer bought a meat pie only?
   (d) Determine whether the events involved are mutually exclusive or inclusive. Give reasons.
   (e) Are these events complementary? Give a reason.

4. Thirty learners were asked to state the sports they enjoyed from swimming (S), tennis (T) and hockey (H). The numbers in each set are shown in the venn diagram. One student is then randomly selected.
   (a) Which events are inclusive? Give reasons.
   (b) Which events are mutually exclusive?
   (c) Which events are complementary?
   (d) What is the probability of selecting a learner who enjoyed either hockey or tennis?



5. In a random experiment it was found that:

$P(A) = 0,25;\ \ P(B) = 0,5\ \text{ and }\ P(A \text{ or } B) = 0,625$

(a) Calculate $P(A \text{ and } B)$

(b) Determine, giving reasons, if events A and B are:

(1) mutually exclusive

(2) inclusive

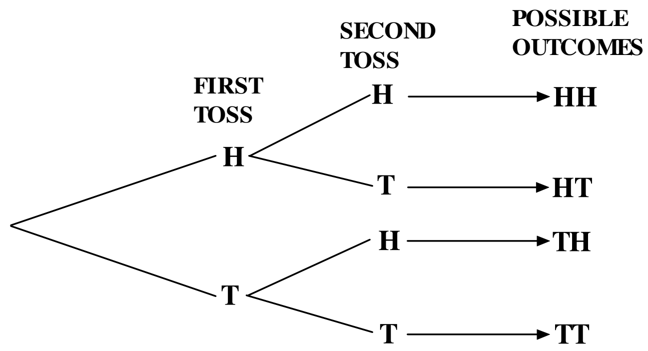
(3) complementary

<u>TREE DIAGRAMS</u>

Tree diagrams are useful tools in determining all possible outcomes of an experiment.

<u>EXAMPLE 1</u>

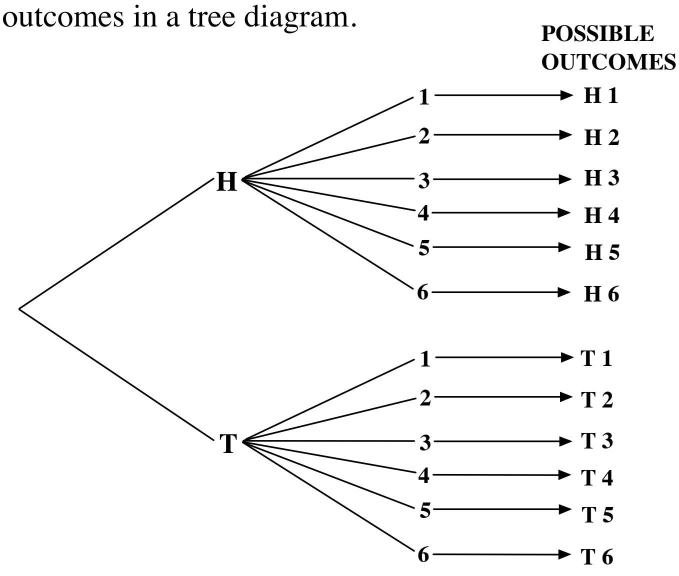
Use a tree diagram to determine the possible outcomes when a coin is tossed twice.



Here the sample space is $S = \{HH, HT, TH, TT\}$.

<u>EXAMPLE 2</u>

Suppose that a coin is tossed and a die is thrown. We can represent the possible outcomes in a tree diagram.



INDEPENDENT AND DEPENDENT EVENTS

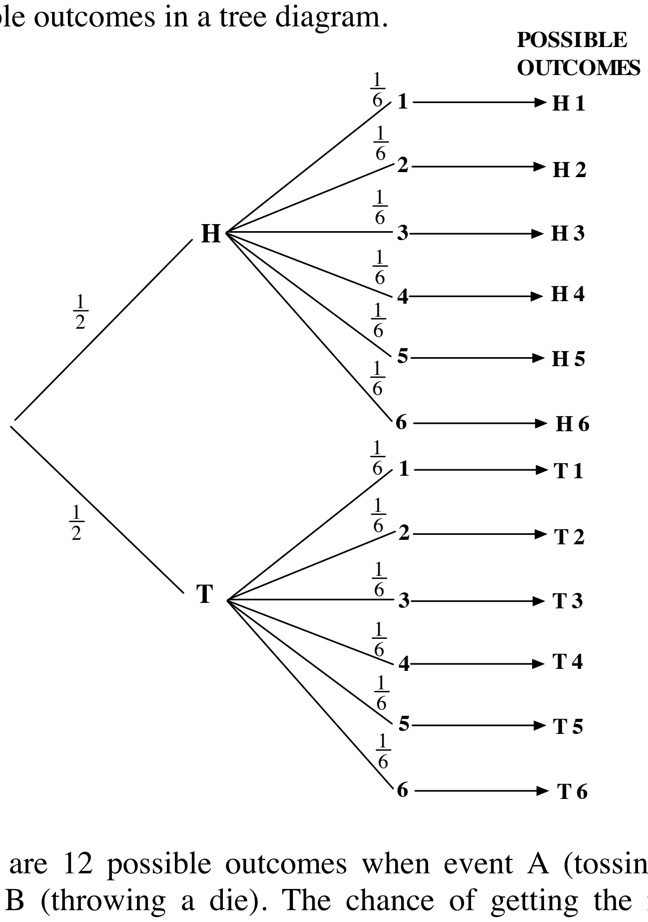
Two successive events A and B are said to be **independent** if the outcomes of the first event do not influence the outcomes of the second event. Event A happens and then is followed by the event B. Suppose that event A is tossing a coin and event B is then throwing a die. The chance of getting a particular number on the die (second event) will not be affected by the chance of first getting a Head or Tail with the coin (first event). Events A and B are clearly independent of each other.

Two successive events are said to be **dependent** if the outcomes of the first event do have an influence on the outcomes of the second event. Suppose that a lunch box contains four sandwiches and two apples. Event A is picking an item of food from the box and then eating it. Event B is then picking an item from the box again and eating it. Clearly event B depends on what happened in event A. If a sandwich was selected in event A, then there will only be 3 sandwiches and two apples left to select from. There will only be 5 items to choose from rather than 6 items, because one sandwich is not replaced. Event A and B are dependent (what happens in event B depends on what previously happened in event A).

INDEPENDENT EVENTS

EXAMPLE 3

Suppose that a coin is tossed and then a die is thrown. We can represent the possible outcomes in a tree diagram.



There are 12 possible outcomes when event A (tossing a coin) is followed by event B (throwing a die). The chance of getting the number 1,2,3,4,5 or 6 is independent of whether you previously first got a Head or Tail. Event A and B are independent.

If we now consider these events in terms of probabilities, some interesting information emerges.

With the tossing of the coin
The probability of getting a Head is 1 chance out of 2 possible outcomes (Heads or Tails). We say that $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$.

In terms of the whole sample space $P(H) = \frac{6}{12}$.

With the throwing of the die
The chance of getting a particular number, say, the number 4 is 1 out of 6 possible outcomes $\{1,2,3,4,5,6\}$. $P(\text{the number } 4) = \frac{1}{6}$.

In terms of the whole sample space, $P(\text{the number } 4) = \frac{2}{12}$.

Note:

- The chance of getting the outcome H4 is 1 out of the 12 possible outcomes. $P(\text{Head and the number } 4) = \frac{1}{12}$.
- The probability of getting a Head is $\frac{1}{2}$, i.e. $P(H) = \frac{1}{2}$ and the probability of getting the number 4 is $\frac{1}{6}$, i.e. $P(4) = \frac{1}{6}$. This can be interpreted as getting $\frac{1}{6}$ of $\frac{1}{2}$ which equals $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.
  This is the same as getting $P(\text{Head and the number } 4) = \frac{1}{12}$.
- We can therefore conclude that $P(H \text{ and } 4) = P(H) \times P(4)$.

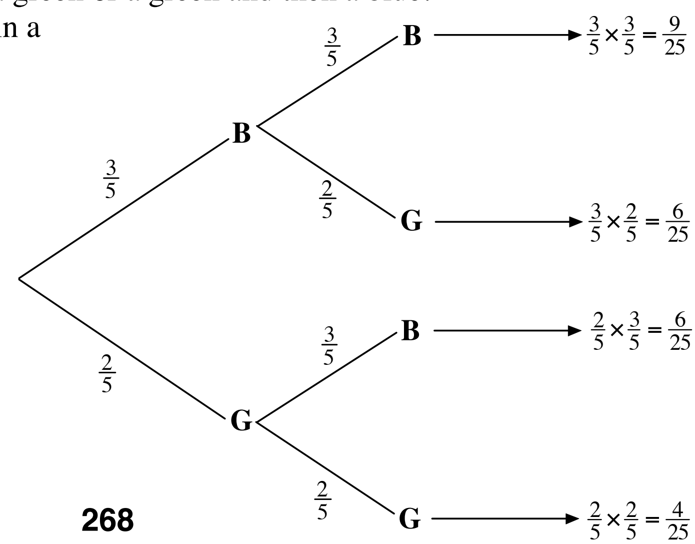
In general, if two events A and B are **independent**, then: $P(A \text{ and } B) = P(A) \times P(B)$. An alternative notation is: $P(A \cap B) = P(A) \times P(B)$.

<u>EXAMPLE 4</u>

A box contains three blue smarties and two green smarties. A smartie is drawn at random and then replaced in the box. Another smartie is then drawn at random and replaced in the box. Determine the probability of:

(a) first drawing a blue smartie and then a green smartie.
(b) first drawing a green smartie and then a blue smartie.
(c) drawing a blue smartie and then another blue smartie.
(d) not drawing a blue smartie on the first or second draw.
(e) drawing a blue and then a green or a green and then a blue.

We first represent the outcomes in a tree diagram.



Notice:

The probability of first drawing a blue smartie is $\frac{3}{5}$. The probability of first drawing a green smartie is $\frac{2}{5}$. The probability of then drawing a blue smartie is $\frac{3}{5}$. The probability of then drawing a green smartie is $\frac{2}{5}$. Because the smarties are replaced after being drawn, it is clear that the second drawing of a smartie is independent of the first draw.

(a) $P(\text{B and G}) = P(B) \times P(G) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

(b) $P(\text{G and B}) = P(G) \times P(B) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$

(c) $P(\text{B and B}) = P(B) \times P(B) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$

(d) $P(\text{not B and not B}) = P(B^{|}) \times P(B^{|}) = \left[\frac{2}{5}\right] \times \left[\frac{2}{5}\right] = \frac{4}{25}$

(e) $P(\text{B and G}) \text{ or } P(\text{G and B}) = P(B) \times P(G) + P(G) \times P(B)$

$$= \frac{3}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{3}{5} = \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$$

Notice that all the probabilities add up to 1:

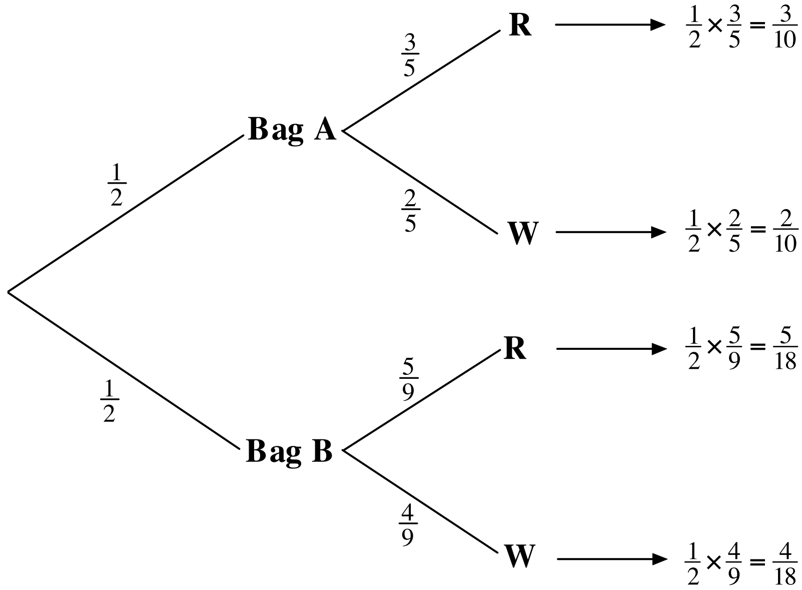
$\frac{9}{25} + \frac{6}{25} + \frac{6}{25} + \frac{4}{25} = \frac{25}{25} = 1$

<u>EXAMPLE 5</u>

Bag A contains 3 red marbles and 2 white marbles. Bag B contains 5 red marbles and 4 white marbles. A bag is chosen at random and then a marble is chosen from that bag and then replaced in the bag.

(a) Determine the probability of getting a red marble.

(b) Determine the probability of getting a white marble.



(a) P(Bag A and Red) or P(Bag B and Red)

$= P(\text{Bag A}) \times P(\text{Red}) + P(\text{Bag B}) \times P(\text{Red})$

$= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9} = \frac{3}{10} + \frac{5}{18} = \frac{26}{45}$

(b) P(Bag A and White) or P(Bag B and White)

$= P(\text{Bag A}) \times P(\text{White}) + P(\text{Bag B}) \times P(\text{White})$

$= \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{9}$

$= \frac{1}{5} + \frac{2}{9}$

$= \frac{19}{45}$

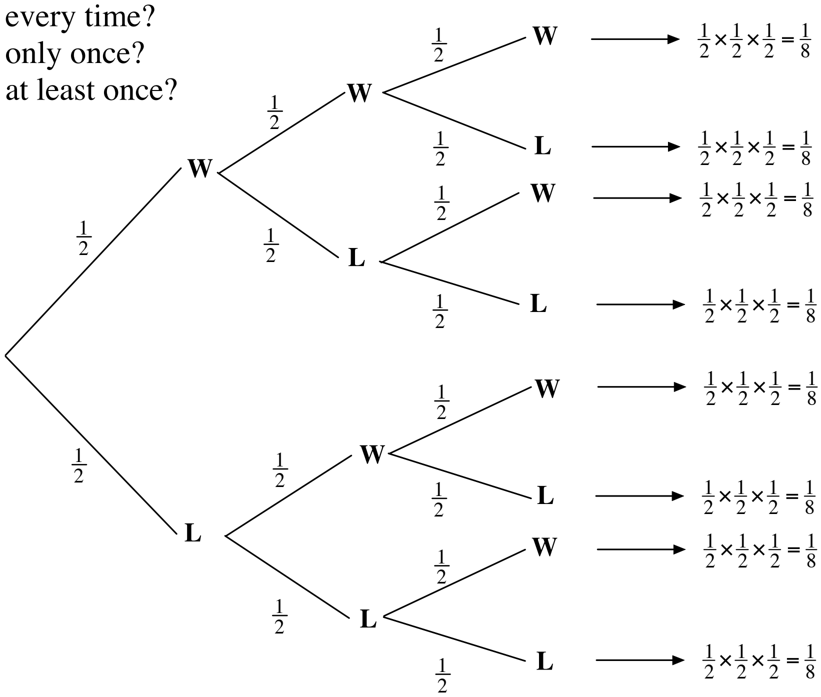
EXAMPLE 6

Consider three consecutive soccer matches. What is the probability that the captain will win the toss:

(a) every time?

(b) only once?

(c) at least once?



(a) $P(W) \times P(W) \times P(W)$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

(b) $= P(W) \times P(L) \times P(L) + P(L) \times P(W) \times P(L) + P(L) \times P(L) \times P(W)$

$= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$= \frac{3}{8}$

(c) P(at least one win)

= P(one or more wins)

= 1 − P(no wins)

$= 1 - P(L) \times P(L) \times P(L)$

$= 1 - \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = 1 - \frac{1}{8} = \frac{7}{8}$

EXAMPLE 7

Three dice are rolled at the same time. Find the probability of not obtaining three sixes landing face up.

P(6 and 6 and 6)

$= P(6) \times P(6) \times P(6)$

$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

$\therefore P\left[\text{not (6 and 6 and 6)}\right]$

$= 1 - P(6 \text{ and } 6 \text{ and } 6)$

$= 1 - \frac{1}{216} = \frac{215}{216}$

EXERCISE 2

1. A bag contains 8 blue marbles and 2 red marbles. One marble is drawn at random and then replaced. A second marble is then drawn and replaced.
   (a) Draw a tree diagram to represent all the possible outcomes.
   (b) What is the probability of obtaining two red marbles?
   (c) What is the probability of obtaining one red marble and one blue marble?
2. A coin is tossed three times.
   (a) Draw a tree diagram to represent all the possible outcomes.
   (b) What is the probability of obtaining the outcome:
   Head then Tail then Tail?
   (c) What is the probability of obtaining the following outcomes:
   Tail then Tail then Head or Head then Tail then Tail?
   (d) What is the probability of obtaining one or more tails?
   (e) What is the probability of obtaining:
   Head then Head then Head or Head then Tail then Head?
3. Sean's lunch box contains four sandwiches and three bananas. He chooses an item of food buts replaces it. He then chooses another item at random and also replaces it.
   (a) Draw a tree diagram.
   (b) Find the probability that he will choose a banana first and then a sandwich.
   (c) Find the probability that he will choose a sandwich first and then a banana.
   (d) Find the probability that he will choose two sandwiches.
   (e) Find the probability that he will choose either a sandwich or banana in any order.
4. Sean now chooses another item from the lunch box. This means that he has made three choices one after the other.
   (a) Draw a tree diagram in this situation.
   (b) Find the probability that he will first choose a banana, then a sandwich and then another sandwich.
   (c) Find the probability that he will choose three bananas.
   (d) Find the probability that he will choose two bananas after his third choice.

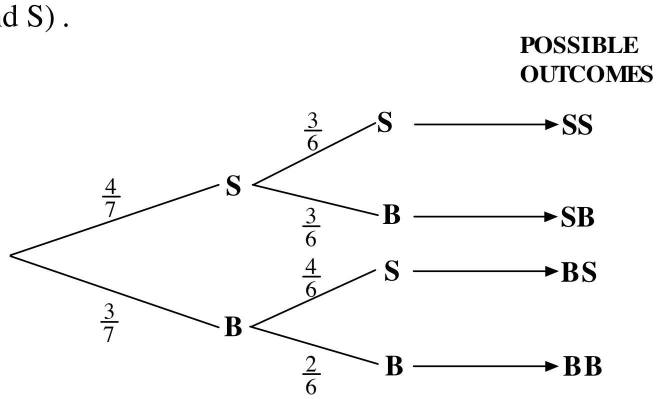
5. Two dice are thrown.
   (a) Draw a tree diagram to represent all the possible outcomes.
   (b) What is the probability of obtaining two number fours?
   (c) What is the probability of obtaining a two and a five?
   (d) What is the probability of obtaining only one two?
   (e) What is the probability of not obtaining two threes landing face up?

DEPENDENT EVENTS

Two successive events are said to be **dependent** if the outcomes of the first event do have an influence on the outcomes of the second event.

EXAMPLE 8

Sean's lunch box contains four sandwiches and three bananas. He chooses an item of food and eats it. He then chooses another item at random and eats it. Find the probability that he will first choose a banana and then a sandwich, i.e. find P(B and S) .



If a sandwich is chosen first and not replaced, then there will only be three sandwiches left to choose from in the next event. However, there will still be three bananas to choose from in the next event. If a banana is chosen first and not replaced, then there will only be two bananas left to choose from in the next event. However, there will still be four sandwiches to choose from in the next event.

Therefore $P(\text{B and S}) = \frac{3}{7} \times \frac{4}{6} = \frac{12}{42} = \frac{2}{7}$

Consider the following:

$P(B).P(S) = \frac{3}{7} \times \frac{4}{7} = \frac{12}{49}$

Clearly, $P(\text{B and S}) \neq P(B).P(S)$

Clearly, for successive **dependent** events A and B: $P(\text{A and B}) \neq P(A).P(B)$

In the previous example, the second event, (S), is restricted by what happened in the first event. The probability that event S occurs once B has occurred is called the **conditional probability** of S given B.

We write this as P(S following B) or P(S/B) .

We say that:
P(B and S) = P(B).P(S/B) = $\frac{3}{7} \times \frac{4}{6} = \frac{2}{7}$

For successive **dependent** events A and B: P(A and B) = P(A).P(B/A)

EXAMPLE 9

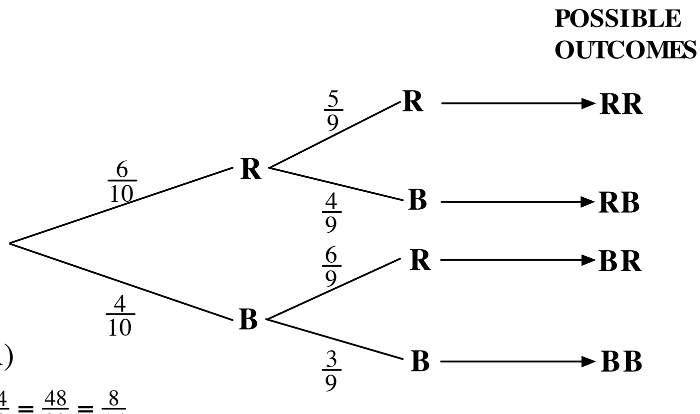
A bag has 6 red and 4 blue marbles. A marble is drawn at random but not replaced. A second marble is then drawn and not replaced. Calculate the following probabilities:
(a) P(first marble drawn is red)
(b) P(both marbles are blue)
(c) P(one marble is red and the other is blue)

Solutions

(a) P(first red) $= \frac{6}{10}$

(b) P(both marbles are blue)
= P(B and B)
$= \frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$

(c) $P\left[(\text{R and B}) \text{ or } (\text{B and R})\right]$
$= P(R) \times P(B) + P(B) \times P(R)$
$= \frac{6}{10} \times \frac{4}{9} + \frac{4}{10} \times \frac{6}{9} = \frac{24}{90} + \frac{24}{90} = \frac{48}{90} = \frac{8}{15}$



EXERCISE 3

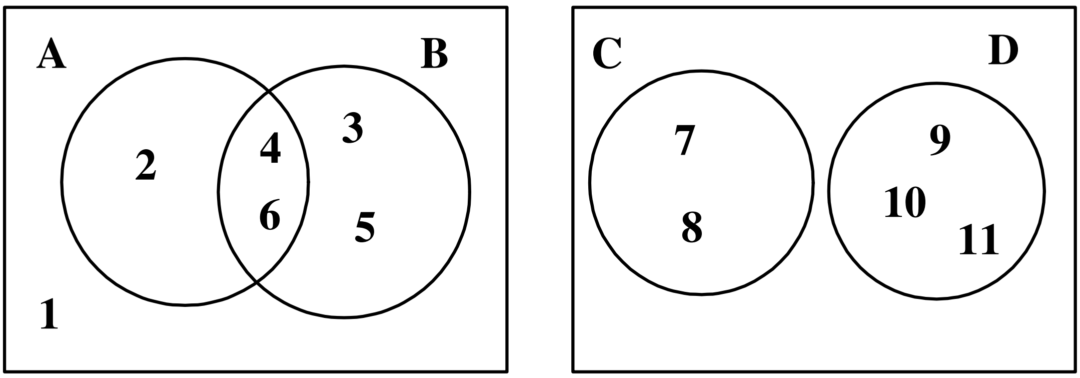
1. There are six green pencils and five purple pencils on a table. James removes a pencil from the table and doesn't replace it. He then removes another pencil.
   Draw a tree diagram.
   Determine the probability that:
   (a) both pencils removed are purple.
   (b) both pencils removed are green.
   (c) a green and a purple pencil is removed.
   (d) at least one green pencil is removed.
2. A school holds a raffle to raise funds for the matric dance. A total of 250 tickets are sold. There are three prizes to be won:
   First prize-Motorbike Second prize-Bicycle Third prize-television
   Michael buys 8 tickets. He could win all three prizes.
   (a) Draw a tree diagram.
   (b) Find the probability that Michael wins no prize.
   (c) Find the probability that Michael wins one prize.
   (d) Find the probability that Michael wins two prizes.
   (e) Find the probability that Michael wins all three prizes.

3. A bag contains 3 white marbles, 4 blue marbles and 3 red marbles.Two marbles are taken out of the bag simultaneously. Draw a tree diagram and then calculate the probability that:
   (a) both marbles are white.
   (b) both marbles are of the same colour.
   (c) both marbles differ in colour.
4. It is winter in England. If the probability that there will be snow tomorrow if it is snowing today is 0,7 and the probability that it will snow tomorrow if it is clear today is 0,4. It is snowing on Saturday. What is the probability that it will snow on Monday?
5. Sean's lunch box contains four sandwiches and three bananas. He chooses an item of food and eats it. He then chooses another item at random and eats it. He then chooses a third item and eats it.
   (a) Draw a tree diagram
   (b) Find the probability that he will first choose a banana, then a sandwich and then another sandwich.
   (c) Find the probability that he will choose three bananas.
   (d) Find the probability that he will have chosen two bananas after his third choice.

<u>VENN DIAGRAMS</u>

<u>EXAMPLE 10</u>

Consider the following Venn diagrams.



Calculate the following:

(a) $P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$ since $A = \{2; 4; 6\}$ and $S = \{1; 2; 3; 4; 5; 6\}$

(b) $P(B) = \frac{4}{6} = \frac{2}{3}$ since $B = \{3; 4; 5; 6\}$ and $S = \{1; 2; 3; 4; 5; 6\}$

(c) $P(\text{not } A) = \frac{3}{6} = \frac{1}{2}$ since not $A = \{1; 3; 5\}$ and $S = \{1; 2; 3; 4; 5; 6\}$

(d) $P(\text{not } B) = \frac{2}{6} = \frac{1}{3}$ since not $B = \{1; 2\}$ and $S = \{1; 2; 3; 4; 5; 6\}$

(e) $A \text{ or } B = \{2 ; 3 ; 4 ; 5 ; 6\}$

(f) $P(A \text{ or } B) = \frac{5}{6}$

(g) $A \text{ and } B = \{4 ; 6\}$

(h) $P(A \text{ and } B) = \frac{2}{6} = \frac{1}{3}$

(i) $\text{not } (A \text{ or } B) = \{1\}$

(j) $P[\text{not } (A \text{ or } B)] = \frac{1}{6}$

(k) $\text{not } (A \text{ and } B) = \{1 ; 2 ; 3 ; 5\}$

(l) $P[\text{not } (A \text{ and } B)] = \frac{4}{6} = \frac{2}{3}$

(m) $A \text{ and not } B = \{2\}$

(n) $P(A \text{ and not } B) = \frac{1}{6}$

(o) $\text{not } A \text{ and } B = \{3 ; 5\}$

(p) $P(\text{not } A \text{ and } B) = \frac{2}{6} = \frac{1}{3}$

(q) $A \text{ or not } B = \{1 ; 2 ; 4 ; 6\}$

(r) $P(A \text{ or not } B) = \frac{4}{6} = \frac{2}{3}$

(s) $\text{not } A \text{ or } B = \{1 ; 3 ; 4 ; 5 ; 6\}$

(t) $P(\text{not } A \text{ or } B) = \frac{5}{6}$

(u) $P(A) + P(A^{|}) = \frac{3}{6} + \frac{3}{6} = 1$

(v) $P(B) + P(B^{|}) = \frac{4}{6} + \frac{2}{6} = 1$

(w) $P(C \text{ or } D) = \frac{5}{5} = 1$

(x) $P(C \text{ and } D) = \frac{0}{5} = 0$

(y) $P(C) + P(D) = \frac{3}{5} + \frac{2}{5} = 1$

EXAMPLE 11

There are 420 learners in a school. 190 learners play soccer, 260 learners play cricket and 60 play both soccer and cricket.

(a) Draw a Venn diagram to represent this information.

(b) Calculate the probability that a learner chosen at random plays soccer or cricket.

(c) Calculate the probability that a learner chosen at random plays neither soccer nor cricket.

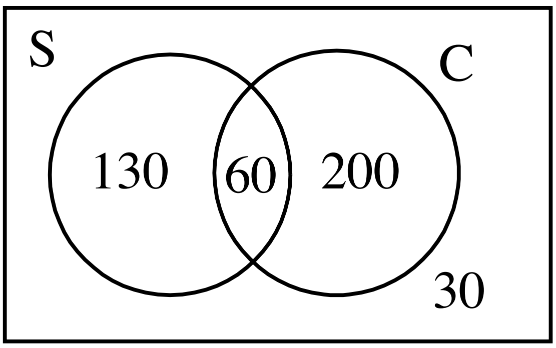
(d) Calculate the probability that a learner chosen at random plays only one of the two sports.

(e) Calculate the probability that a learner chosen at random does not play both sports.

(f) Calculate the probability that a learner chosen at random plays both sports.

Solutions

(a)



(b) P(S or C) $= \frac{130+60+200}{420} = \frac{390}{420} = \frac{13}{14}$

(c) P(not S or not C) $= \frac{30}{420} = \frac{1}{14}$

(d) P(only one sport) $= \frac{130}{420} + \frac{200}{420} = \frac{330}{420} = \frac{11}{14}$

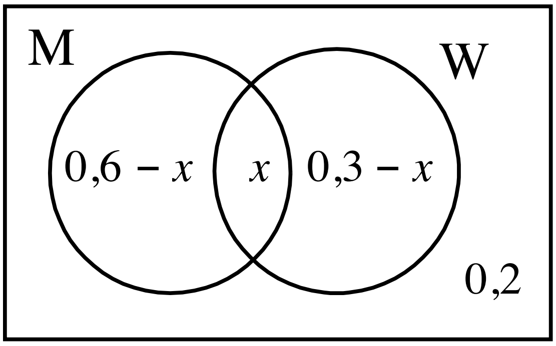
(e) P(not playing both sports) $= \frac{130}{420} + \frac{200}{420} + \frac{30}{420} = \frac{360}{420} = \frac{12}{14} = \frac{6}{7}$

(f) P(S and C) $= \frac{60}{420}$

EXAMPLE 12

The probability that a person drinks milk is 0,6. The probability of drinking water is 0,3. The probability of drinking neither is 0,2. Calculate the probability of:
(a) drinking both milk and water.
(b) drinking only one of the two drinks.

Solutions



(a) Let x be in $M \cap W$

We know that

$P(M \text{ or } W) + P[\text{not in } (M \text{ or } W)] = 1$

$\therefore P(M) + P(W) - P(M \text{ and } W) + P[\text{not in } (M \text{ or } W)] = 1$

$0,6 + 0,3 - x + 0,2 = 1$ (all probabilities must add up to 1)

$\therefore -x = 1 - 0,2 - 0,3 - 0,6$

$\therefore -x = -0,1$

$\therefore x = 0,1$

$\therefore P(M \text{ and } W) = 0,1$

(b) P(drinks only one) $= (0,6 - 0,1) + (0,3 - 0,1) = 0,5 + 0,2 = 0,7$

EXAMPLE 13

The manager of a hotel in Pretoria recorded the number of guests sitting down for breakfast, lunch and supper on a particular day. Of the 300 guests, 29 did not arrive for any of the three meals.
153 were at breakfast
161 were at lunch
145 were at supper
95 were at beakfast and lunch
80 were at lunch and supper
52 were at supper but did not arrive for the other two meals.
70 were at all three meals

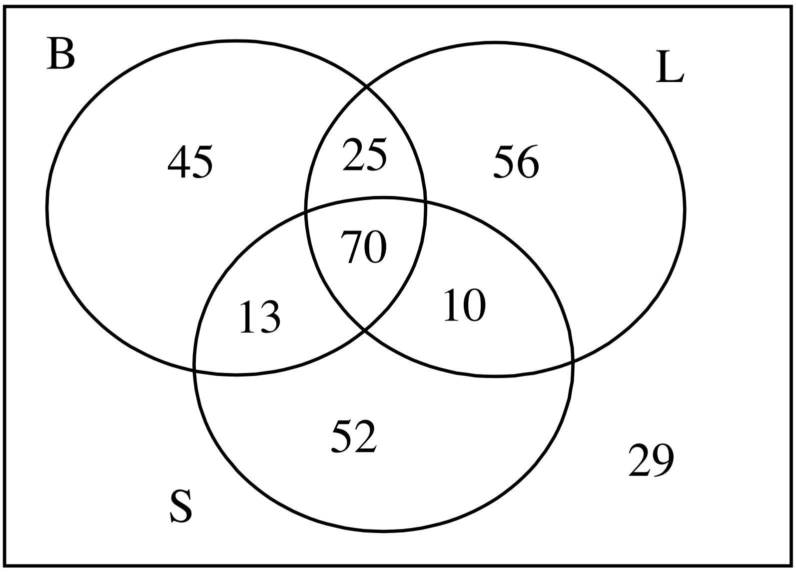
(a) Draw a venn diagram.

(b) Calculate the probability that a guest chosen at random will have:
   (1) been at both breakfast and lunch, but not supper.
   (2) been at both breakfast and lunch.
   (3) been at breakfast only.
   (4) been at one or more of the meals.

(a) The best way to do this is to start with the intersection, i.e. 70 guests and then develop from there.

Solutions

(a)



(b) (1) been at both breakfast and lunch, but not supper.

$$\frac{25}{300} = \frac{1}{12}$$

(2) been at both breakfast and lunch.

$$\frac{95}{300} = \frac{19}{60}$$

(3) been at breakfast only.

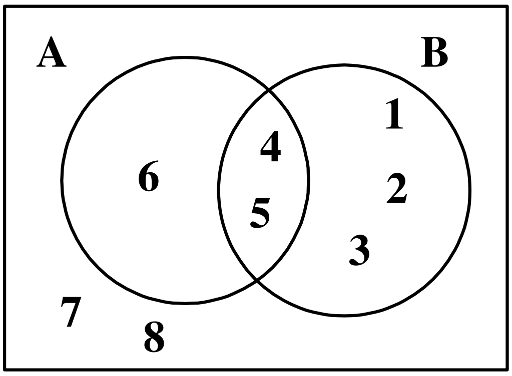
$$\frac{45}{300} = \frac{3}{20}$$

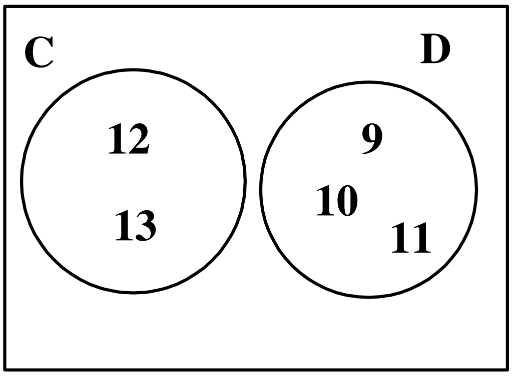
(4) been at one or more of the meals.

$$1 - \frac{29}{300} = \frac{271}{300}$$

EXERCISE 4

1. Consider the following Venn diagrams.





Calculate the following:

(a)	P(A)	(b)	P(B)	(c)	P(not A)
(d)	$P(B')$	(e)	A or B	(f)	P(A or B)
(g)	A and B	(h)	P(A and B)	(i)	not (A or B)
(j)	$P[\text{not (A or B)}]$	(k)	not (A and B)	(l)	$P[\text{not (A and B)}]$
(m)	A and not B	(n)	P(A and not B)	(o)	not A and B
(p)	P(not A and B)	(q)	A or B'	(r)	$P(A \text{ or } B')$
(s)	A' or B	(t)	$P(A' \text{ or } B)$	(u)	P(C or D)
(v)	P(C and D)	(w)	$P(C) + P(C')$	(x)	$1 - P(C)$

2. The probability that Joe will today see a movie is 0,7. The probability that he will go to a restaurant is 0,8. The probability of him seeing a movie and going to a restaurant is 0,6. Determine the probability that:
   (a) he doesn't go to a movie or a restaurant.
   (b) he only goes to a movie.
   (c) he only goes to a restaurant.
   (d) doesn't go to a movie.
   (e) doesn't go to a restaurant.
   (f) he goes to either one or the other.

3. If it is given that $P(A) = 0{,}35$; $P(B) = 0{,}8$ and $P(A \text{ and } B) = 0{,}25$,
   Use a venn diagram and a formula to determine:
   (a) P(A or B) (b) $P(A' \text{ and } B)$ (c) $P(A' \text{ or } B)$
   (d) $P(A \text{ and } B)'$ (e) $P(A \text{ or } B)'$

4. There are 130 Grade 11 learners at a particular school. 50 learners take Biology and 68 learners take Maths. 32 learners take both subjects.
   (a) Draw a Venn Diagram to illustrate the given data.
   (b) Hence calculate the probability that a learner in Grade 11, chosen at random:
      (1) takes Biology.
      (2) takes Maths and Biology.

5. In a Grade 11 class there are 30 students. 16 of them are girls. There are 7 girls and 6 boys with blue eyes. A student is selected at random to be the class monitor.
   (a) Draw a Venn diagram to represent the information.
   (b) Find the probability that the class monitor is a girl.
   (c) Find the probability that the class monitor is a boy with blue eyes.
   (d) Find the probability that the class monitor has not got blue eyes.
   (e) Find the probability that the class monitor is a girl and has not got blue eyes.
6. A group of 80 athletes entered the 100m, 200m and 400m sprints as follows:
   6 entered all three events.
   21 entered none of these events.
   10 entered the 100m and 200m
   11 entered the 200m and 400m
   Of the 21 who entered the 100m, 10 entered nothing else.
   27 entered the 400m
   (a) Represent the above situation using a Venn Diagram.
   (b) How many athletes entered the 200m event?
   (c) What is the probability of an athlete, selected at random, running in at least two of the sprint events?
7. There are 200 registered delegates for an upcoming seminar on Financial Management to be held at the Sandton Convention Centre. The most popular courses in the seminar are usually Share Market Basics, Retirement Planning and Debt Management. There are other less popular courses, which some of the delegates attend. The following information was extracted from the registration forms:
   107 delegates registered for Share Market Basics (S)
   90 delegates registered for Retirement Planning (R)
   63 delegates registered for Debt Management (D)
   35 delegates registered for Retirement Planning and Share Market Basics
   23 delegates registered for Share Market Basics and Debt Management
   15 delegates registered for Share Market Basics and Retirement Planning and Debt Management.
   190 delegates registered for Share Market Basics or Retirement Planning or Debt Management.
   x delegates registered for Debt Management and Retirement Planning, but not Share Market Basics
   (a) Draw a venn diagram.
   (b) Calculate the value of x.
   (c) How many of the delegates have not registered for any of Share Market Basics, Retirement Planning and Debt Management?
   (d) How many of the delegates have registered for Retirement Planning and Debt Management, but not Share Market Basics?
   (e) What is the probability that a delegate selected at random registered for at least two of the following courses:
   Share Market Basics, Retirement Planning and Debt Management?

CONTINGENCY TABLES (ENRICHMENT)

Contingency tables are a way of classifying sample observations according to two or more identifiable characteristics (for example, gender and gym use). A contingency table shows a count or number of observations grouped by specific characteristics.

EXAMPLE 14

A Durban gym recorded the number of members (men and women) who use or don't use the gym on a regular basis. The results are recorded in the following contingency table:

	Men	**Women**	**Total**
Use the gym regularly	70	150	220
Don't use the gym regularly	110	40	150
Total	180	190	370

Calculate the probability that a member selected at random from the sample of 370 members:

(a) uses the gym regularly.

(b) uses the gym regularly given that the member is a woman.

(c) doesn't use the gym regularly given that the member is a man.

Solutions

(a) There are a total of 370 members in the sample of which 220 use the gym. The probability of selecting a woman at random from this sample is:

$$\frac{220}{370} = 0{,}59$$

(b) The probability of selecting at random a member that is a regular gym user given that the member is also a woman, is:

$$\frac{150}{190} = 0{,}79 \quad \text{(sample space is reduced to women only)}$$

(c) The probability of selecting at random a member that is not a regular gym user given that the member is also a man, is:

$$\frac{110}{180} = 0{,}61 \quad \text{(sample space is reduced to men only)}$$

EXERCISE 5

1. A school investigated the number of learners who arrived at school on time, arrived late or were absent during a particular week. The results are recorded in the following table. There were 100 learners.

	Boys	**Girls**	**Total**
On time	40	25	65
Late	18	7	25
Absent	7	3	10
Total	65	35	100

   (a) Calculate the probability of a learner arriving late for school.
   (b) Calculate the probability of a learner arriving on time.
   (c) Calculate the probability of a learner being absent if the learner is a boy.
   (d) Calculate the probability of a learner arriving on time if the learner is a girl.
   (e) Calculate the probability of a learner arriving late or being absent.
   (f) Calculate the probability of a learner arriving late or being absent if the learner is a boy.

2. The hair colour of thirty learners was recorded in the following contingency table:

	Boys	**Girls**	**Total**
Black	6	2	8
Brown	8	4	12
Blonde	4	6	10
Total	18	12	30

   Calculate the probability that a learner, chosen at random:
   (a) will have black hair.
   (b) will have brown hair given that a girl is chosen.
   (c) will have blonde hair given that a boy is chosen.
   (d) will be a girl given that the hair colour chosen is blonde.

3. A study of speeding fines issued recently and drivers who use car phones was conducted by the Johannesburg Traffic Department. The following data was recorded.

	Speeding fines	**No speeding fines**	**Total**
Car phone user	25	280	305
Not a car phone user	45	405	450
Total	70	685	755

   Calculate the probability that a person selected at random:
   (a) is a car phone user.
   (b) had no speeding fines.
   (c) is a car phone user given that the person had a speeding fine.
   (d) had no speeding fine given that the person did not use a car phone.

REVISION EXERCISE

1. A carpark contains 10 BMW cars and 12 Toyota cars. Two cars are stolen at different times, being selected at random. What is the probability that the stolen cars are:
   (a) both BMW's
   (b) one BMW and one Toyota (in any order)?
2. A study was done to determine the efficacy of three different drugs A, B and C in relieving headache pain. Over the period covered by the study, 80 patients were given the chance to use all three drugs. The following results were obtained:
   40 reported relief from drug A
   35 reported relief from drug B
   40 reported relief from drug C
   21 reported relief from both drugs A and C
   18 reported relief from drugs B and C
   68 reported relief from at least one of the drugs
   7 reported relief from all three drugs
   (a) Record this information in a Venn diagram.
   (b) How many of the patients got relief from none of the drugs?
   (c) How many of the patients got relief from drugs A and B but not C?
   (d) What is the probability that a randomly selected subject got relief from at least two of the drugs?
3. (a) Assuming that the probability of a female birth is 0,49, construct a tree diagram to show the possible outcomes of the sexes of children in a family of 3 children.
   (b) For a family with 3 children, use the tree diagram to find the probability of obtaining:
      (1) exactly one girl
      (2) at least one girl
      (3) at least one child of each sex.
4. A recent survey investigated the causes of business bankruptcy. The data collected showed that 37,5% of business bankruptcies are caused by lack of capital; 50% are caused by lack of business ability and 31,25% can be attributed to neither of these causes.
   (a) Draw a Venn diagram representing this data and hence find the percentage of business bankruptcies caused by both lack of capital and lack of business ability.
   (b) Explain fully whether the two events are independent or not.
5. Two dice are thrown. Consider the following events in the sample space:
   Event A: the numbers add up to 8
   Event B: the number are both even
   (a) Draw a Venn diagram to represent these events.
   (b) Determine whether these events are:
      (1) inclusive or mutually exclusive
      (2) dependent or independent

(Taken from old Add Maths past papers)

SOME CHALLENGES

1. In a class of 40 learners, 25 take Biology, 16 take Geography, 15 take Science and 14 take Computer Science. Also, 6 learners take both Geograpy and Computer Science. All learners who take Computer Science also take Biology, 9 learners take Science only and 7 take Biology only. All learners take either Biology or Science.
   (a) Draw a Venn Diagram.
   (b) If a learner is chosen at random, find the probability of that learner taking:
      (1) Biology and Geography but not Computer Science
      (2) Biology or Geography
      (3) Computer Science or Science
   (c) If a learner taking Biology is selected at random, determine whether it is more likely or less likely that the learner also takes
      (1) Geography (2) Computer Science

2. Two dice are thrown. The possible outcomes as well as the sum of the two numbers are shown in the following table.

	1	**2**	**3**	**4**	**5**	**6**
1	(1 ; 1) sum = 2	(2 ; 1) sum = 3	(3 ; 1) sum = 4	(4 ; 1) sum = 5	(5 ; 1) sum = 6	(6 ; 1) sum = 7
2	(1 ; 2) sum = 3	(2 ; 2) sum = 4	(3 ; 2) sum = 5	(4 ; 2) sum = 6	(5 ; 2) sum = 7	(6 ; 2) sum = 8
3	(1 ; 3) sum = 4	(2 ; 3) sum = 5	(3 ; 3) sum = 6	(4 ; 3) sum = 7	(5 ; 3) sum = 8	(6 ; 3) sum = 9
4	(1 ; 4) sum = 5	(2 ; 4) sum = 6	(3 ; 4) sum = 7	(4 ; 4) sum = 8	(5 ; 4) sum = 9	(6 ; 4) sum = 10
5	(1 ; 5) sum = 6	(2 ; 5) sum = 7	(3 ; 5) sum = 8	(4 ; 5) sum = 9	(5 ; 5) sum = 10	(6 ; 5) sum = 11
6	(1 ; 6) sum = 7	(2 ; 6) sum = 8	(3 ; 6) sum = 9	(4 ; 6) sum = 10	(5 ; 6) sum = 11	(6 ; 6) sum = 12

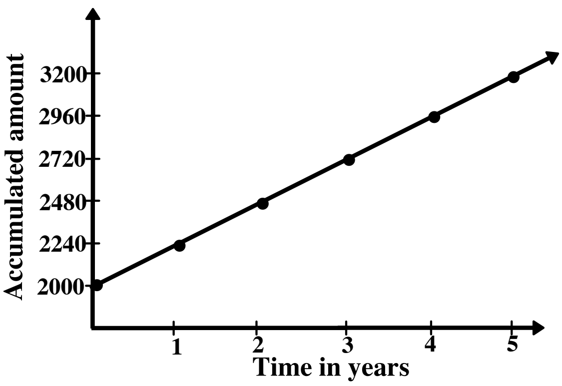
   (a) Calculate the probability of obtaining at least one number 4.
   (b) Calculate the probability of obtaining a sum of 9 or more.
   (c) Calculate the probability of obtaining a sum of less than 9.
   (d) Calculate the probability of getting the number 4 given a sum of 9 or more.
   (e) Calculate the probability of getting the number 4 given the number 3.
   (f) Calculate the probability of getting at least one 2 or one 3 given that the sum of the two numbers is 6 or 7.

CHAPTER 10 – FINANCIAL MATHEMATICS

REVISION OF SIMPLE AND COMPOUND INTEREST

SIMPLE INTEREST

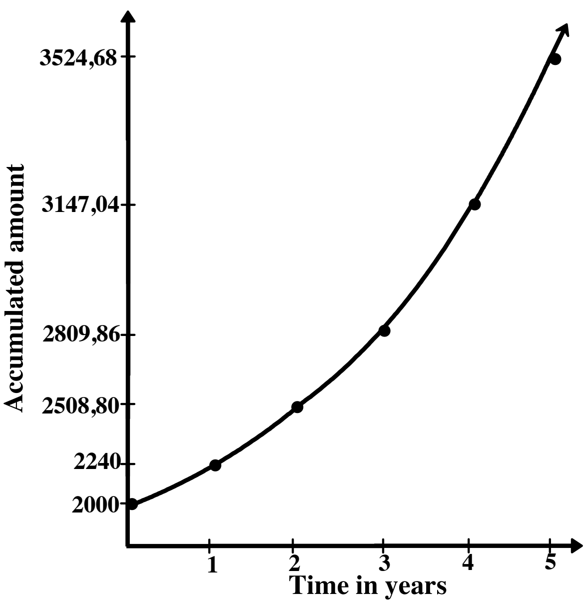
Consider an amount of R2 000 invested at 12% per annum **simple** interest:
Accumulated amount after one year:
$A_1 = 2000 + 0{,}12 \times 2000 = \text{R}2240$
(Interest received is R240)
Accumulated amount after two years:
$A_2 = 2240 + 0{,}12 \times 2000 = \text{R}2480$
(Interest received is R240)
Accumulated amount after three years:
$A_3 = 2480 + 0{,}12 \times 2000 = \text{R}2720$
(Interest received is R240)
Accumulated amount after four years:
$A_4 = 2720 + 0{,}12 \times 2000 = \text{R}2960$
(Interest received is R240)
Accumulated amount after five years:
$A_5 = 2960 + 0{,}12 \times 2000 = \text{R}3200$
(Interest received is R240)



Notice: The interest received each year is calculated using the original amount invested (R2 000). This means that the interest received each year will always be the same (R240). The graph of this relationship is a **linear function**.

COMPOUND INTEREST

Consider an amount of R2 000 invested at 12% per annum **compound** interest:
Accumulated amount after one year:
$A_1 = 2000 + 0{,}12 \times 2000$
$= \text{R}2240$
(Interest received is R240)
Accumulated amount after two years:
$A_2 = 2240 + 0{,}12 \times 2240$
$= \text{R}2508{,}80$
(Interest received is R268,80)
Accumulated amount after three years:
$A_3 = 2508{,}80 + 0{,}12 \times 2508{,}80$
$= \text{R}2809{,}86$
(Interest received is R301,06)
Accumulated amount after four years:
$A_4 = 2809{,}86 + 0{,}12 \times 2809{,}86$
$= \text{R}3147{,}04$



(Interest received is R337,18)
Accumulated amount after five years:

$$A_5 = 3147{,}04 + 0{,}12 \times 3147{,}04$$
$$= R3524{,}68$$

(Interest received is R377,64)

Notice:

The interest received at the end of the first year is the same as that received using the simple interest approach. However, at the end of the second year, the interest received is higher when using the compound interest approach because interest for the second year was calculated using the accumulated amount of the first year (R2240). This will apply to each successive year and therefore the accumulated amount each year will increase exponentially. The graph of this relationship will therefore be an **exponential function**.

Formula for Simple Interest

$$\mathbf{A = P(1 + \mathit{in})}$$

Formula for Compound Interest

$$\mathbf{A = P(1 + \mathit{i})^{\mathit{n}}}$$

P = present value of the investment or loan (original amount at the beginning)

A = accumulated amount or future value of the investment or loan after n periods

n = time period

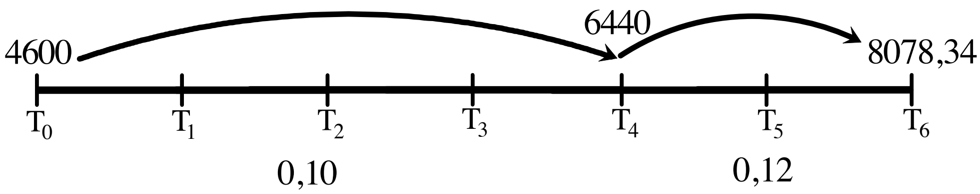
r = interest rate as a percentage

i = interest rate as a decimal

$$i = \frac{r}{100} \qquad r = 100 \times i$$

EXAMPLE 1

Sibongile invests R4600 for four years at 10% per annum simple interest. Thereafter, she invests the accumulated amount for another two years at 12% per annum compound interest. Calculate how much money she will have saved at the end of the six year period.



For the first four years:

$A = P(1 + in)$

$\therefore A = 4600(1 + 0{,}10 \times 4)$

$\therefore A = R6440$

For the remaining two years:

$A = P(1+i)^n$

$\therefore A = 6440(1+0{,}12)^2$

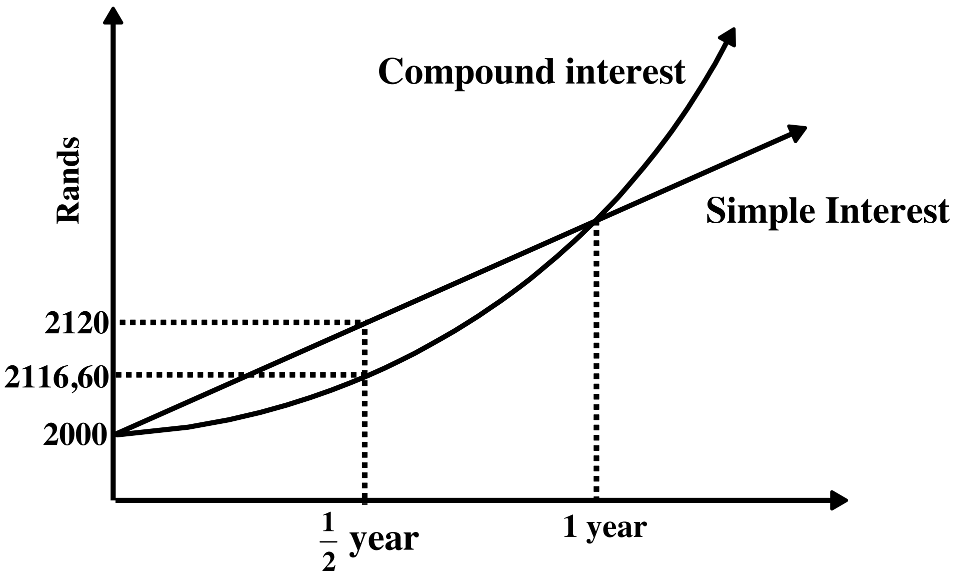
$\therefore A = R8078{,}34$

You could have done the calculation more efficiently in one line as follows:

$A = 4600(1+0{,}10\times 4)\,.\,(1+0{,}12)^2$

$\therefore A = R8078{,}34$

<u>The graphical relationship between simple and compound interest</u>



During the first year, the simple interest growth on the R2 000 will be greater than the compounded growth on the R2 000.

For example, consider the growth of the R2 000 for the first six months (half a year):

Using simple interest: $A = 2000(1+0{,}12\times 0{,}5) = R2120$

Using compound interest: $A = 2000(1+0{,}12)^{0{,}5} = R2116{,}60$

At the end of the first year, the future value of R2 000 using simple interest and compound interest will be the same.

Using simple interest: $A = 2000(1+0{,}12\times 1) = R2240$

Using compound interest: $A = 2000(1+0{,}12)^1 = R2240$

After the first year, the compound interest growth will be greater than the simple interest growth. Consider, for example, the growth after 2 years:

Using simple interest: $A = 2000(1+0{,}12\times 2) = R2480$

Using compound interest: $A = 2000(1+0{,}12)^2 = R2508{,}80$

<u>EXERCISE 1</u> **(REVISION OF GRADE 10 FINANCIAL MATHS)**

1. Stephanie invests R5 000 for 6 years. Find the future value of her investment if the interest she receives is:
   (a) 12% p.a. simple interest.
   (b) 12% p.a. compound interest.

2. Marc saved an amount of money and it grew to R15 000 over a period of seven years. Calculate the amount of money originally invested if the interest received was:
   (a) 13% p.a. simple interest.
   (b) 13% p.a. compound interest.
3. R90 000 is invested at 10% p.a. simple interest for 4 years. Thereafter, the total amount is re-invested in a different financial institution at 12% p.a. compound interest for three more years. What is the future value of the investment at the end of the seven-year period?
4. Nolene invests R9 000 at an interest rate of 6% per annum compounded annually for a period of four years. Thereafter, the interest rate changes to 7% per annum compounded annually for a further two years. Calculate the future value of the investment at the end of the six-year period.
5. In five years' time Peter wants to have saved R50 000 in order to visit his friend who lives in Ireland. He manages to receive an interest rate of 14% per annum simple interest. How much must he invest now in order to achieve this goal?
6. Janet invests R3 000 and it accumulates to R4 000 over a period of two years. What simple interest rate will she need to receive in order to achieve this?
7. Pauline borrows money from a bank in order to finance a marketing business. The bank charges her an interest rate of 14% p.a. compounded annually. Calculate the amount she originally borrowed, if she pays off the loan in six years' time with a payment of R500 000.
8. R6 000 is invested for four years and grows to R7 000. Find the interest rate if interest is compounded annually.

DEPRECIATION

In many situations, equipment can lose its value over a given time period. Motor vehicles or computers lose value over time due to wear and tear or because of becoming outdated. When this happens, we say that the equipment is depreciating in value over time.

Book value: is the value of equipment at a given time after depreciation has taken place.

Scrap value: is the book value of the equipment at the end of its useful life.

There are two types of depreciation:

Linear depreciation and **reducing balance depreciation**.

LINEAR DEPRECIATION (STRAIGHT LINE DEPRECIATION)

With linear depreciation, equipment is depreciated by a percentage of its original value. It works in the same way as simple interest, but the value decreases rather than increases as with simple interest.

Consider, for example, the value of a motor vehicle after six years if it depreciates at a rate of 10% per annum on a linear depreciation scale.

$A_1 = 200\ 000 - 0{,}10 \times 200\ 000$

$\therefore A_1 = R180\ 000$ (The car's value depreciated by R20 000)

$A_2 = 180\ 000 - 0{,}10 \times 200\ 000$

$\therefore A_2 = R160\ 000$ (The car's value depreciated by a further R20 000)

$A_3 = 160\ 000 - 0{,}10 \times 200\ 000$

$\therefore A_3 = R140\ 000$ (The car's value depreciated by a further R20 000)

$A_4 = 140\ 000 - 0{,}10 \times 200\ 000$

$\therefore A_4 = R120\ 000$ (The car's value depreciated by a further R20 000)

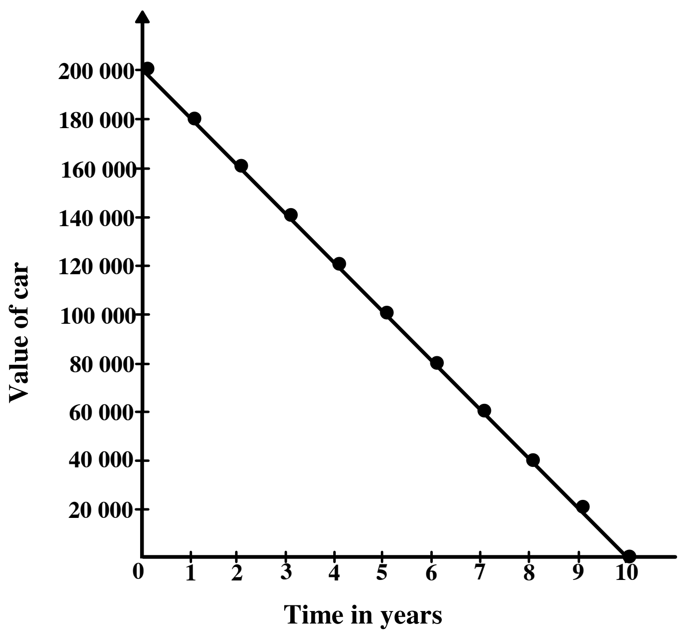
$A_5 = 120\ 000 - 0{,}10 \times 200\ 000$

$\therefore A_5 = R100\ 000$ (The car's value depreciated by a further R20 000)

$A_6 = 100\ 000 - 0{,}10 \times 200\ 000$

$\therefore A_6 = R80\ 000$ (The car's value depreciated by a further R20 000)

Notice: The motor vehicle's value depreciates by the same amount each year (R20 000 each year). After 6 years, the car is worth R80 000. We can represent this information on a graph as follows:



Notice: The graph of this depreciation is a ***linear function*** (straight-line graph). The value of the car will eventually be zero, because the graph will cut the horizontal axis where the value of the car as read from the vertical axis will be zero.

A useful formula to calculate linear depreciation is:

$A = P(1 - in)$
where
A = book or scrap value
P = present value
i = depreciation rate
n = time period

Consider, for example, the value of a car after 6 years if it depreciates at a rate of 10% per annum using linear depreciation.

$A = P(1 - in)$

$\therefore A = 200\ 000(1 - 0{,}10 \times 6)$

$\therefore A = R80\ 000$

REDUCING-BALANCE DEPRECIATION

With reducing-balance depreciation, equipment is depreciated by a percentage of its previous value. It works in the same way as compound interest, but the value decreases rather than increases as with compound interest.
Consider, for example, the value of a motor vehicle after 4 years if it depreciates at a rate of 10% per annum using reducing balance depreciation.

$A_1 = 200\ 000 - 0{,}10 \times 200\ 000$

$\therefore A_1 = R180\ 000$ (The car's value depreciated by R20 000)

$A_2 = 180\ 000 - 0{,}10 \times 180\ 000$

$\therefore A_2 = R162\ 000$ (The car's value depreciated by a further R18 000)

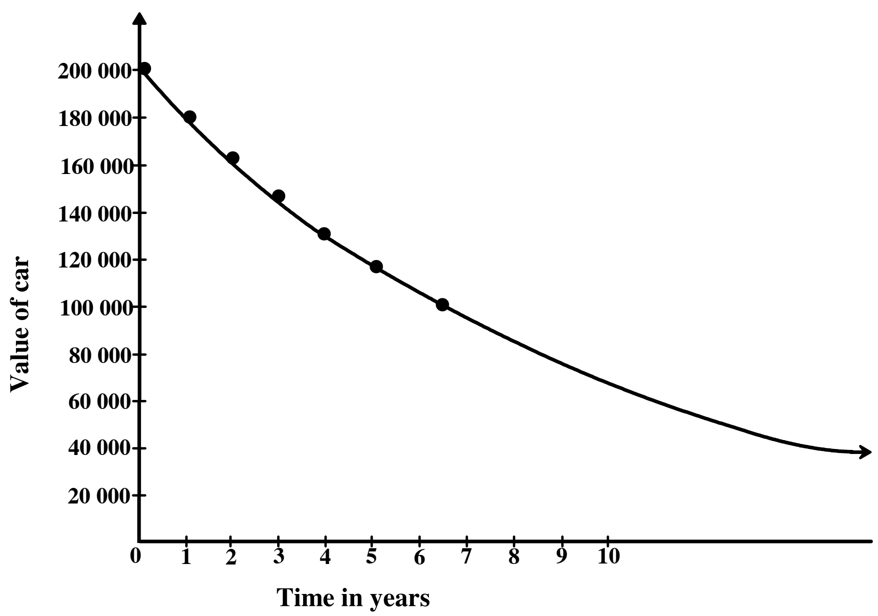
$A_3 = 162\ 000 - 0{,}10 \times 162\ 000$

$\therefore A_3 = R145\ 800$ (The car's value depreciated by a further R16 200)

$A_4 = 145\ 800 - 0{,}10 \times 145\ 800$

$\therefore A_4 = R131\ 220$ (The car's value depreciated by a further R14 580)

Notice: The car's value depreciates by different lesser amounts each year. After 4 years, the car is worth R131 220. Depreciation is calculated on the reducing balance each year.
We can represent this information on a graph as follows:



Notice: The graph of this depreciation is a decreasing ***exponential function***. The car will always have some value as the years progress. This is because the exponential graph never cuts the horizontal axis so as to produce a value of zero rands (as read off the vertical axis).

A useful formula to calculate reducing balance depreciation is:

$A = P(1-i)^n$
where
A = book or scrap value
P = present value
i = depreciation rate
n = time period

Consider, for example, the value of a car after 4 years if it depreciates at a rate of 10% per annum using reducing balance depreciation.

$A = P(1-i)^n$

$\therefore A = 200\ 000(1-0{,}10)^4$

$\therefore A = R131\ 220$

EXAMPLE 2

Joshua wants to sell his car in five years' time. The rate of depreciation is 14% per annum and the car's current value is R60 000. Calculate the book value of the car in five years' time, if depreciation is based on:
(a) the straight-line method.
(b) the reducing-balance method.

Solutions

(a) $A = 60\ 000(1-0{,}14\times 5)$
$\therefore A = R18\ 000$

(b) $A = 60\ 000(1-0{,}14)^5$
$\therefore A = R28\ 225{,}62$

EXAMPLE 3

Calculate the original price of a computer if its depreciated value after seven years is R900 and the rate of depreciation was 12% per annum calculated using:
(a) the straight-line method.
(b) the reducing-balance method.

Solutions

(a) $A = P(1-in)$
$\therefore 900 = P(1-0{,}12\times 7)$
$\therefore 900 = P(0{,}16)$
$\therefore \frac{900}{0{,}16} = P$
$\therefore P = R5625$

(b) $A = P(1-i)^n$
$\therefore 900 = P(1-0{,}12)^7$
$\therefore 900 = P(0{,}88)^7$
$\therefore \frac{900}{(0{,}88)^7} = P$
$\therefore P = R2202{,}24$

EXAMPLE 4

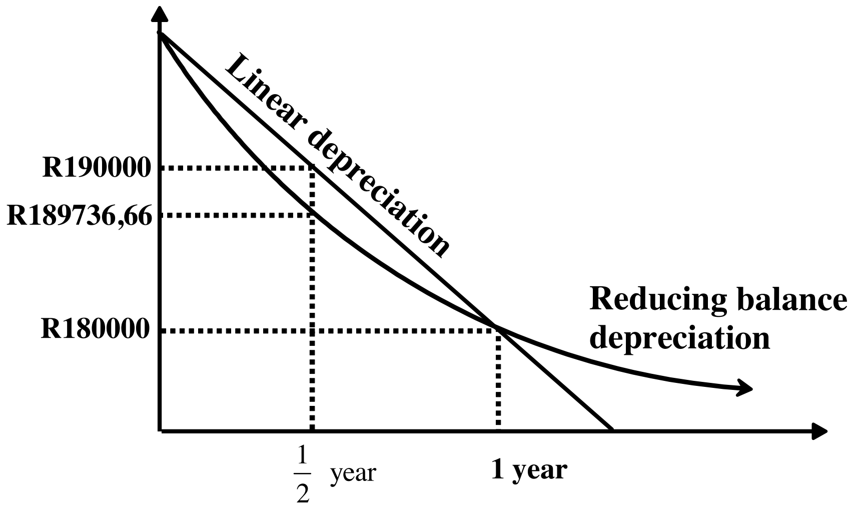
A laptop cost R9 000 and, after four years, has a scrap value of R2 000. Find the annual depreciation rate if it is calculated using:
(a) the straight line method. (b) the reducing balance method.

Solutions

(a)
$$2000 = 9000(1-4i)$$
$$\therefore \frac{2000}{9000} = (1-4i)$$
$$\therefore 4i = 1-\frac{2000}{9000}$$
$$\therefore 4i = \frac{7}{9}$$
$$\therefore i = \frac{7}{36}$$
$$\therefore i = 0{,}19\dot{4}$$
$$\therefore r = 19{,}4\%$$

(b)
$$2000 = 9000(1-i)^4$$
$$\therefore \frac{2000}{9\,000} = (1-i)^4$$
$$\therefore \left(\frac{2}{9}\right)^{\frac{1}{4}} = 1-i$$
$$\therefore i = 1-\left(\frac{2}{9}\right)^{\frac{1}{4}} = 0{,}313410952$$
$$\therefore r = 31{,}3\%$$

The graphical relationship between linear and reducing-balance depreciation



During the first year the car will have a higher depreciated value with linear depreciation as compared with reducing balance depreciation.
For example, consider the depreciation of a car costing R200 000 for the first six months (half a year) at a depreciation rate of 10% per annum:
Using linear depreciation: $A = 200000(1-0{,}10\times 0{,}5) = R190000$
Using reducing balance: $A = 200000(1-0{,}10)^{0{,}5} = R189736{,}66$
At the end of the first year the depreciated value using linear and reducing balance depreciation will be the same.
Using linear depreciation: $A = 200000(1-0{,}10\times 1) = R180000$
Using reducing balance: $A = 200000(1-0{,}10)^{1} = R180\,000$

After the first year, the car will have a higher depreciated value with reducing balance as compared with linear depreciation. Consider, for example, the depreciation after 2 years:

Using linear depreciation: $A = 200000(1 - 0{,}10 \times 2) = R160000$

Using reducing balance: $A = 200000(1 - 0{,}10)^2 = R162000$

EXERCISE 2

1. Nelson buys a motor car for R160 000. The car depreciates at a rate of 11% p.a. What is the car worth after 6 years if depreciation is calculated using:
   (a) the straight-line method. (b) the reducing-balance method.
2. Calculate the original price of a computer if its depreciated value after 5 years is R1200 and the rate of depreciation was 13% per annum calculated using:
   (a) the straight-line method? (b) the reducing-balance method?
3. A photocopy machine costs R40 000 and has a scrap value of R8 000 after 8 years. Find the annual rate of depreciation if it is calculated using:
   (a) the straight-line method. (b) the reducing-balance method.
4. A motor vehicle currently has a book value of R6 000. The rate of depreciation was 14% per annum using the reducing balance method. Calculate the original price of the motor vehicle, if it was bought 5 years ago.
5. The office computers for a small business are currently worth R40 000. Calculate the value of these computers after 6 years, if the rate of depreciation is 16% per annum calculated on a linear basis.
6. A school buys a photocopying machine for R200 000. Calculate the scrap value of the machine after 10 years if the rate of depreciation is 14% per annum calculated on the reducing balance scale.
7. A hotel manager installs kitchen equipment in his hotel costing R120 000. He allows for depreciation on a linear basis over a period of 8 years. Calculate the rate of annual depreciation if the value of the equipment is reduced to zero at the end of the 8 year period.
8. A supermarket purchases a large refrigerator for R1 000 000. The manager expects to replace the refrigerator at the end of 4 years. At that stage the refrigerator will have no resale value at all.
   (a) What annual rate of depreciation will be used if the refrigerator is to be discarded at the end of 4 years?
   (b) Write down the depreciated value of the refrigerator year by year for the 4 years.

DIFFERENT COMPOUNDING PERIODS

Up to now, we have dealt with annual interest rates. However, it can happen that interest can be quoted per annum but calculated over different time periods during a year. Interest can be calculated:

⇨ **annually**: once per year (usually at the end of the year)
⇨ **semi-annually (half-yearly)**: twice per year (every six months)
⇨ **quarterly**: four times per year (every three months)
⇨ **monthly**: twelve times per year (every one month)
⇨ **daily**: 365 times per year (excluding leap years)

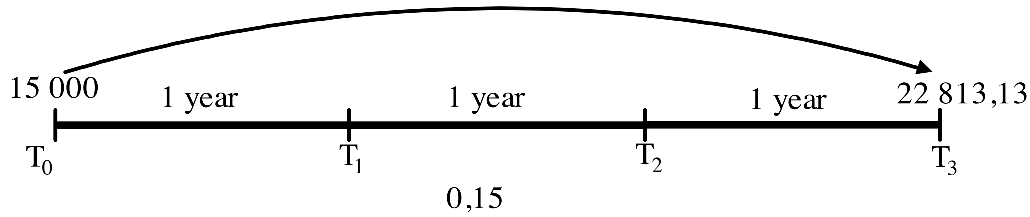
EXAMPLE 5

Calculate the future value of an investment of R15 000 after three years at an interest rate of 15% per annum compounded:

(a) annually (b) half-yearly
(c) quarterly (d) monthly
(e) daily (excluding leap years)

Solutions

(a)

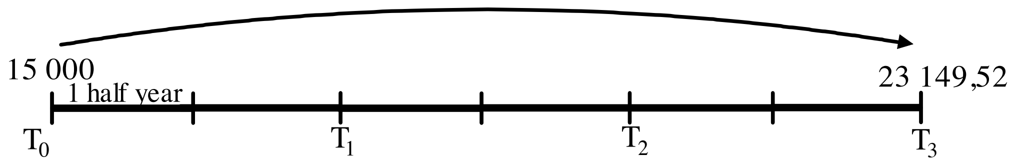


$A = 15\ 000(1+0,15)^3$

$\therefore A = 15\ 000(1,15)^3$

$\therefore A = R22\ 813,13$

(b) 2 half years in one year $\times 3$ years $= 6$ half years



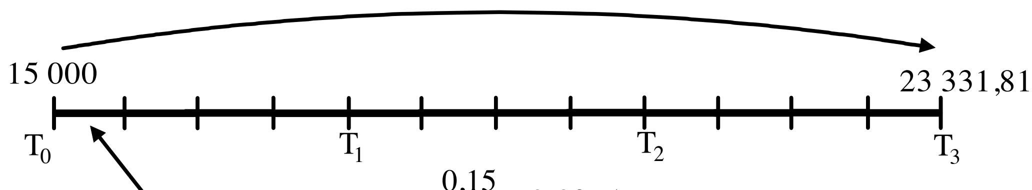
$$\frac{0,15}{2} = 0,075$$

$$A = 15\ 000\left(1+\frac{0,15}{2}\right)^6$$

$\therefore A = 15\ 000(1,075)^6$

$\therefore A = R23\ 149,52$

(c) 4 quarters in one year $\times 3$ years $= 12$ quarters



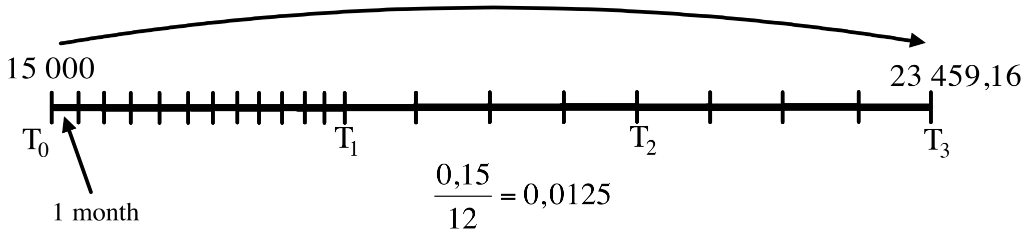
$$\frac{0,15}{4} = 0,0375$$

$$A = 15\ 000\left(1+\frac{0,15}{4}\right)^{12}$$

$\therefore A = 15\ 000(1,0375)^{12}$

$\therefore A = R23\ 331,81$

(d) 12 months in one year $\times 3$ years $= 36$ months



$$A = 15\ 000\left(1 + \frac{0,15}{12}\right)^{36}$$

$$\therefore A = 15\ 000(1,0125)^{36}$$

$$\therefore A = R23\ 459,16$$

(e) There are 365 days in a year (excluding leap years).
Therefore, there are $365 \times 3 = 1095$ days in three years.
The daily interest rate will be $\frac{0,15}{365}$

$$A = 15\ 000\left(1 + \frac{0,15}{365}\right)^{1095}$$

$$\therefore A = R23\ 522,51$$

Notice: **As the number of compounding periods increase during a year, the greater the accumulated amount at the end.**

EXAMPLE 6

What amount must be invested for 2 years at an interest rate of 10% per annum compounded half-yearly in order to receive R10 000?

Solution

$$10\ 000 = P\left(1 + \frac{0,10}{2}\right)^{2 \text{ years} \times 2 \text{ half years}}$$

$$\therefore 10\ 000 = P(1,05)^4$$

$$\therefore \frac{10\ 000}{(1,05)^4} = P$$

$$\therefore 10\ 000(1,05)^{-4} = P$$

$$\therefore P = 10\ 000(1,05)^{-4}$$

$$\therefore P = R8227,02$$

Notice that the formula to find P when given A is as follows:

$$P = A(1+i)^{-n}$$

The previous example could therefore have been done directly:

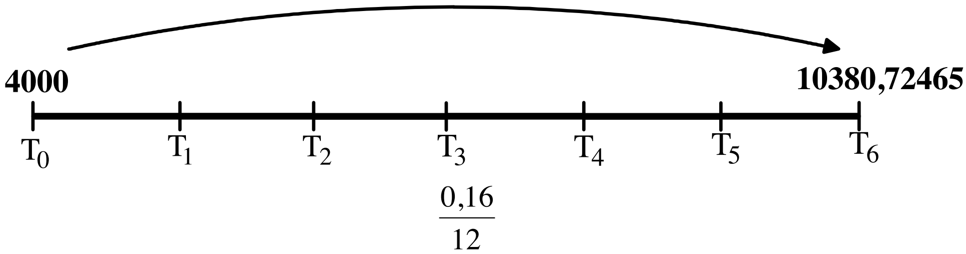
$$P = 10\ 000\left(1+\frac{0{,}10}{2}\right)^{-4} = \text{R}8227{,}02$$

The following example will clearly illustrate the use of the two formulae.

EXAMPLE 7

- Using the formula $A = P(1+i)^n$

  Suppose that R4 000 is invested for 6 years at an interest rate of 16% per annum compounded monthly. Calculate the value of the investment at the end of the six year period.

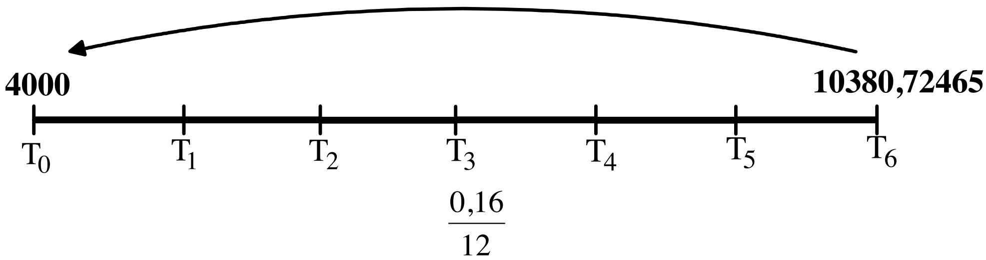


$$A = 4000\left(1+\frac{0{,}16}{12}\right)^{72} = 10380{,}72465$$

Notice: The exponent is **<u>positive</u>** and the movement on the time-line is from **<u>left to right</u>** because interest on the R4 000 is growing.

- Using the formula $P = A(1+i)^{-n}$

  Suppose that R10380,72465 is received if a certain amount of money was invested 6 years ago. The interest rate was 16% per annum compounded monthly. What was the original amount invested?



$$P = 10380{,}72465\left(1+\frac{0{,}16}{12}\right)^{-72} = 4000$$

Notice: The exponent is **<u>negative</u>** and the movement on the time-line is from **<u>right to left</u>** because interest on the R10380,72465 is being removed.

There are therefore two formulae available:

$A = P(1+i)^n$	(to find A)
$P = A(1+i)^{-n}$	(to find P)

EXERCISE 3

1. Pauline invests R20 000. Calculate the future value of her investment in six years' time, if the interest rate is 15% per annum compounded:
   (a) annually (b) half-yearly (c) quarterly
   (d) monthly (e) daily (excluding leap years)
2. David invests R8 000 in a savings account which pays 8% per annum compounded monthly. Calculate the value of his investment in ten years' time.
3. You invest R60 000 for 15 years. Calculate the future value of your savings if you receive an interest rate of 12% per annum compounded annually. Now calculate the future value of your investment had the interest rate been 12% per annum compounded monthly. Which option produced the highest return?
4. R100 000 is invested for five years at an interest rate of 16% per annum compounded quarterly. Thereafter the accumulated amount is re-invested for a further six years at an interest rate of 15% per annum compound semi-annually. Calculate the value of the investment at the end of the eleven-year period.
5. Determine which of the following two savings options is a better investment over a period of one year, if the interest is calculated at:
   A. 15% per annum compounded monthly
   B. 17% per annum compounded quarterly
6. Joseph invested an amount of money six years ago. Now, after six years, it is worth R1 200 000. The interest rate for the savings period was 18% per compounded monthly. What was the amount that was originally invested six years ago?
7. Mark wants to save up for an overseas trip in five years' time. He needs to have saved R120 000. If he receives an interest rate of 8% per annum compounded monthly, how much money must he save now?
8. Alice buys a new car and agrees to pay it in full in 3 years' time with a once-off payment of R300 000. If the interest rate is 18% per annum compounded half-yearly, calculate the purchase price of her car.

NOMINAL AND EFFECTIVE INTEREST RATES

Nominal interest rates

A nominal rate is an annual rate which financial institutions quote. This interest rate does not take into consideration the effect of different compounding periods, which are shorter than the annual period. For example, 15% per annum compounded monthly is a nominal rate. The annual rate is 15% but the fact that the interest is compounded monthly means that the accumulated amount at the end will be higher. The quoted annual rate of 15% will yield a lower accumulated amount than 15% compounded monthly.

Consider the following example to illustrate this.
R2 000 invested for one year at 18% per annum without monthly compounding:

$$A = 2000(1+0{,}18)^1$$

$$\therefore A = R2360$$

R2 000 invested for one year at 18% per annum with monthly compounding:

$$A = 2000\left(1+\frac{0{,}18}{12}\right)^{12}$$

$$\therefore A = R2391{,}24$$

Clearly, the monthly compounding yielded a higher accumulated amount than the quoted annual rate of 18% without compounding.

It is possible to determine an annual rate compounded annually that will yield the same accumulated amount as the annual rate compounded monthly.
The original amount invested was R2 000. The money accumulated to R2 391,24 using monthly compounding. Therefore, the money gained was R391,24. As a percentage, this is $\frac{391{,}24}{2000} = 19{,}562\%$, which is higher than the quoted 18%.

Notice:
If we now use this rate of 19,562% per annum compounded annually, the accumulated amount will be the same as the 18% per annum compounded monthly:

$$A = 2000(1+0{,}19562)^1$$

$$\therefore A = R2391{,}24$$

We call 19,562% the ***annual effective rate***. It is equivalent to 18% per annum compounded monthly because it produces the same accumulated amount as the 18% per annum compounded monthly. The quoted 18% is the ***nominal rate.***

Effective interest rates

Annual effective interest rates are therefore the equivalent annual rates that yield the same accumulated amount as rates with different compounding periods.
Annual effective rates are higher than quoted nominal rates. We refer to the annual effective rate by means of the notation i_{eff}.
If the interest rate is quoted as 18% per annum compounded monthly, we have a nominal rate because the quoted period and the compounding period are different. However, if this interest rate is quoted as $\frac{18\%}{12} = 1{,}5\%$ per **month** compounded **monthly**, we are dealing with what is called an **effective rate per period** because the stated period and compounding period are the same. In calculations involving the nominal rate, it is absolutely essential to work with the effective rate per period. This is clear from the section dealing with different compounding periods. Therefore, if the nominal rate is 18% per annum compounded monthly, you would use the **effective monthly rate** of $\frac{0{,}18}{12} = 0{,}015$ in your calculation.
We now explore a useful formula to convert a nominal rate to its equivalent annual effective rate.

In the previous example:

$$2000(1+0{,}19562)^1 = 2000\left(1+\frac{0{,}18}{12}\right)^{12} \quad \text{(both equal R2391,24)}$$

$$\therefore 2000(1+i_{\text{eff}}) = 2000\left(1+\frac{0{,}18}{12}\right)^{12}$$

$$\therefore (1+i_{\text{eff}}) = \left(1+\frac{0{,}18}{12}\right)^{12}$$

We can therefore create a formula that helps to calculate the annual effective rate when given a nominal rate or vice versa. The formula can be written as follows:

$$\boxed{1+i_{\text{eff}} = \left(1+\frac{i_{\text{nom}}}{n}\right)^{n}}$$

where: i_{eff} = effective rate (annual)

i_{nom} = nominal rate

n = number of compoundings per year

EXAMPLE 8

(a) Convert a nominal rate of 18% per annum compounded monthly to an annual effective rate.

(b) Convert an annual effective rate of 13,5% per annum, to a nominal rate per annum compounded semi-annually.

Solutions

(a) $1+i_{\text{eff}} = \left(1+\frac{0{,}18}{12}\right)^{12}$

$\therefore i_{\text{eff}} = (1{,}015)^{12} - 1 = 0{,}195618171$

$\therefore r_{\text{eff}} = 19{,}6\%$

(b) $1+i_{\text{eff}} = \left(1+\frac{i_{\text{nom}}}{n}\right)^{n}$

$\therefore (1+0{,}135) = \left(1+\frac{i_{\text{nom}}}{2}\right)^{2}$

$\therefore 1{,}135 = \left(1+\frac{i_{\text{nom}}}{2}\right)^{2}$

$\therefore (1{,}135)^{\frac{1}{2}} = 1+\frac{i_{\text{nom}}}{2}$

$\therefore (1{,}135)^{\frac{1}{2}} - 1 = \frac{i_{\text{nom}}}{2}$

$\therefore 0{,}06536378763 = \frac{i_{\text{nom}}}{2}$

$\therefore 0{,}1307275753 = i_{\text{nom}}$

$\therefore r = 13{,}1\%$ per annum compounded half yearly

EXAMPLE 9

Michael invests R40 000 for five years at 16% per annum compounded monthly.

(a) Calculate the future value of the investment using the nominal rate.
(b) Convert the nominal rate of 16% per annum compounded monthly to the equivalent effective rate (annual).
(c) Now use the annual effective rate to show that the same accumulated amount will be obtained as when using the nominal rate.

Solutions

(a)
$$A = 40\,000\left(1+\frac{0,16}{12}\right)^{60}$$
$$\therefore A = R88\ 552,28$$

(b)
$$1+i_{\text{eff}} = \left(1+\frac{0,16}{12}\right)^{12}$$
$$\therefore i_{\text{eff}} = \left(1+\frac{0,16}{12}\right)^{12} - 1$$
$$\therefore i_{\text{eff}} = 0,1722707983$$

(c)
$$A = 40\,000\left(1+0,1722707983\right)^{5}$$
$$\therefore A = 40\ 000(1,1722707983)^{5}$$
$$\therefore A = R88\ 552,28$$

EXERCISE 4

1. Convert the following nominal rates to equivalent annual effective rates:
   (a) 13% per annum compounded half-yearly.
   (b) 15% per annum compounded quarterly.
   (c) 11% per annum compounded monthly.
   (d) 14% per annum compounded daily.
2 (a) Convert an annual effective rate of 13,2% per annum to a nominal rate per annum compounded quarterly.
   (b) Convert an annual effective rate of 14,5% per annum to a nominal rate per annum compounded half-yearly.
   (c) Convert an annual effective rate of 10,5% per annum to a nominal rate per annum compounded monthly.
3. Tamara invests R24 000 at 14% per annum compounded quarterly for a period of twelve years.
   (a) Calculate the future value of the investment using the nominal rate.
   (b) Convert the nominal rate of 14% per annum compounded quarterly to the equivalent effective rate (annual).
   (c) Use the annual effective rate to show that the same accumulated amount will be obtained as when using the nominal rate.
4. Michelle inherited R500 000 and deposited the money into a savings account for a period of six years. The accumulated amount at the end of the six year period was R650 000. Calculate the interest rate paid in each of the following cases:
   (a) The annual effective rate.
   (b) The nominal rate per annum compounded monthly.

5. You have R4 000 to invest in a bank. Different banks offer the following nominal interest rates:

Bank A: 18% per annum compounded monthly
Bank B: 17% per annum compounded quarterly
Bank C: 16% per annum compounded half-yearly
Bank D: 17% per annum compounded daily

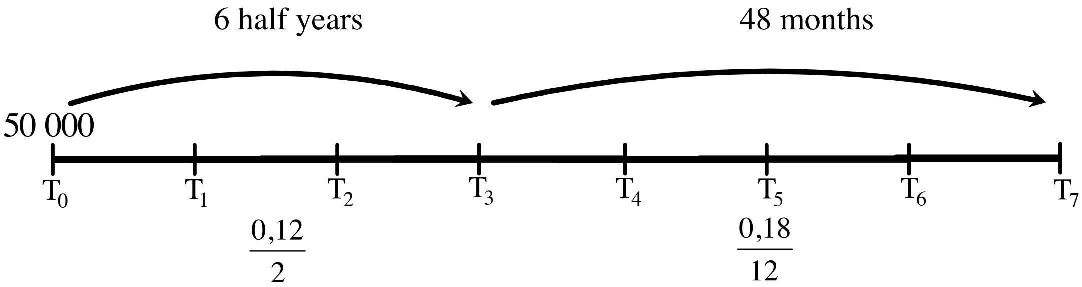
Which bank would you invest your money with? Motivate your answer with appropriate calculations.

CALCULATIONS INVOLVING CHANGING INTEREST RATES

Time lines are useful when dealing with more complicated problems, such as interest rate changes, or when several deposits are made into a savings account or withdrawals are made from the account.

EXAMPLE 10

R50 000 is deposited into a savings account. The interest rate for the first three years of the savings is 12% per annum compounded half-yearly. Thereafter, the interest rate changes to 18% per annum compounded monthly. The money is left to grow for a further four years. Calculate the future value of the investment at the end of the seven-year savings period.



The way to calculate the future value of the investment at T_7 is to grow R50 000 through two interest rates. You first grow the money through the 12% interest rate to obtain the future value at T_3. You then grow the money further through the 18% interest rate in order to obtain the future value at T_7.

Method 1

At T_3: $50000\left(1+\frac{0,12}{2}\right)^6 = 70925,95561$

At T_7: $70925,95561.\left(1+\frac{0,18}{12}\right)^{48} = \text{R}144\ 935,65$

Method 2 **(Highly recommended)**

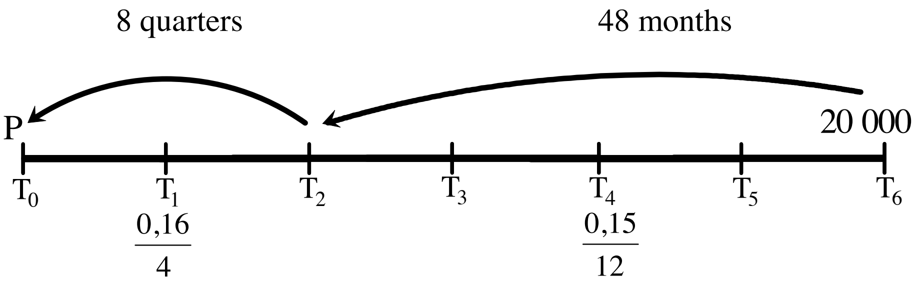
$$A = 50000\left(1+\frac{0,12}{2}\right)^6 . \left(1+\frac{0,18}{12}\right)^{48}$$

$$\therefore A = 50000(1,06)^{16}.(1,015)^4$$

$$\therefore A = \text{R}144\ 935,65$$

EXAMPLE 11

Jerome invests a certain sum of money for six years at 16% per annum compounded quarterly for the first two years and 15% per annum compounded monthly for the remaining term. The future value of the investment at the end of the 6 year period is R20 000. How much did he originally invest?



Method 1

At T_2:

$$20\,000 = P\left(1+\frac{0{,}15}{12}\right)^{48}$$

$$\therefore \frac{20\,000}{\left(1+\frac{0{,}15}{12}\right)^{48}} = P$$

$$\therefore P = 11017{,}12977$$

At T_0:

$$11017{,}12977 = P\left(1+\frac{0{,}16}{4}\right)^{8}$$

$$\therefore \frac{11017{,}12977}{\left(1+\frac{0{,}16}{4}\right)^{8}} = P$$

$$\therefore P = R8050{,}11$$

Method 2 **(Using the formula for P)**

We can use the formula for P to do the calculations as follows:

Present value at T_2: $20\,000\left(1+\frac{0{,}15}{12}\right)^{-48} = 11017{,}12977$

Present value at T_0 $P = 11017{,}12977\left(1+\frac{0{,}16}{4}\right)^{-8} = R8050{,}11$

Method 3 **(shortcut)**

$$P = 20\,000\left(1+\frac{0{,}15}{12}\right)^{-48}.\left(1+\frac{0{,}16}{4}\right)^{-8}$$

$$\therefore P = R8050{,}11$$

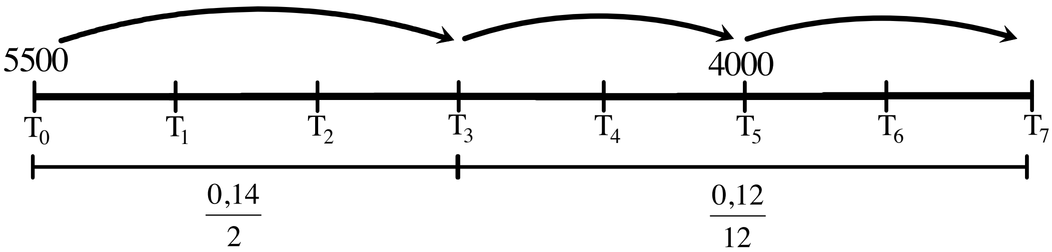
EXERCISE 5

1. Refilwe deposits R18 000 into a savings account. The interest rate for the first three years is 10% per annum compounded half-yearly. Thereafter, the interest rate changes to 8% per annum compounded quarterly. Calculate the value of the investment at the end of the tenth year.
2. Belinda invests R22 000 for a period of eight years into a unit trust fund. During the first five years, the interest rate is 15% per annum compounded monthly. The interest rate then changes to 20% per annum compounded semi-annually. Calculate the future value of her investment at the end of the eight year period.
3. Mpho deposits R90 000 into a savings account paying 13% per annum compounded monthly. After five years, the interest rate increases by 2%. Three years later, the interest rate decreases by 1%. Calculate the value of her investment after ten years.
4. Sibongile deposits R6 000 into a bank account for a period of 12 years. The interest rate for the first seven years is 8% per annum compounded monthly. For the next five years, the interest rate changes to 10% per annum compounded half-yearly.
   (a) Convert the nominal rates to annual effective rates.
   (b) Use the effective rates and calculate the future value of the savings at the end of the twelve-year period.
5. Siphiwe deposits a certain amount in a savings account. It grows to an amount of R13 000 after seven years. The interest rate during the first four years is 9% per annum compounded annually and for the remaining three years is 12% per annum compounded monthly. How much is this amount?
6. Malibongwe wants to have saved R10 000 000 in eight years' time. How much must he invest now if the interest rate for the first six years will be 15% per annum compounded monthly and 20% per annum compounded quarterly for the remaining two years?
7. Theresa wants to save for an overseas trip in three years' time. She will need to have saved R50 000 for the trip. The interest rate during the first year will be 14% per annum compounded quarterly. For the remaining two years, the interest rate will be 11% per annum compounded monthly. What must she invest now in order to receive R50 000 in three years' time?
8. Neeran received a monetary gift from his father. He decides to invest this money in a saving account in order to buy a motor car when he matriculates in four years' time. The expected cost of the motor car in four years' time is R100 000. Suppose that the interest rate during the first two years of the savings period is 14% per annum compounded monthly. For the remaining two years, suppose that the interest rate changes to 13% per annum compounded half-yearly.
   (a) By using the nominal rates, calculate the amount of money Neeran received as a birthday gift.
   (b) Convert the nominal rates to effective annual rates.
   (c) By using the effective rates, calculate the amount of money Neeran received as a birthday gift. What do you notice?

ADDITIONAL PAYMENTS AND WITHDRAWALS

EXAMPLE 12

Brenda deposits R5500 into a savings account. Five years later, R4 000 is added to the savings. The interest rate for the first three years is 14% per annum compounded semi-annually. Thereafter, the interest rate changes to 12% per annum compounded monthly. Calculate the future value of the savings at the end of the seventh year.



Method 1

Value at T_3:

$$5500\left(1+\frac{0,14}{2}\right)^{6}$$
$$=8254,016935$$

Value at T_5:

$$8254,016935\left(1+\frac{0,12}{12}\right)^{24}+4000$$
$$=14480,41129$$

Value at T_7:

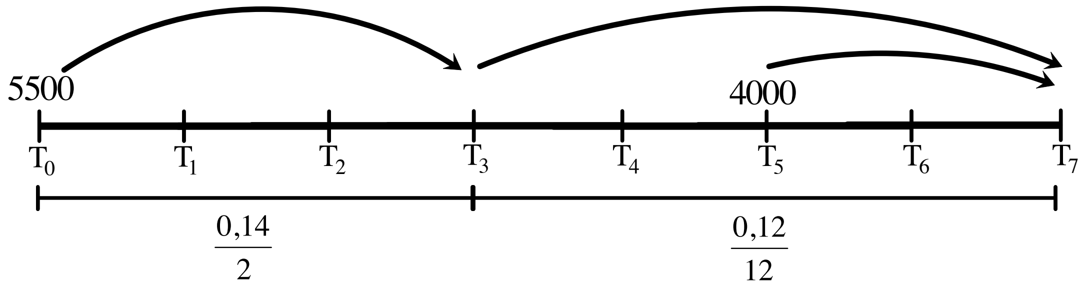
$$14480,41129\left(1+\frac{0,12}{12}\right)^{24}$$
$$=\text{R}18386,28$$

Method 2

$$A=\left\{5500\left(1+\frac{0,14}{2}\right)^{6}\left(1+\frac{0,12}{12}\right)^{24}+4000\right\}\left(1+\frac{0,12}{12}\right)^{24}$$

$$A=\text{R}18\ 386,28$$

Method 3 **(Recommended)**

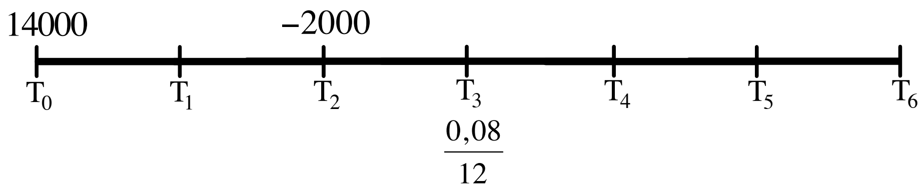


$$A = 5500\left(1+\frac{0,14}{2}\right)^{6}\left(1+\frac{0,12}{12}\right)^{48} + 4000\left(1+\frac{0,12}{12}\right)^{24}$$

$A = \text{R}18\ 386{,}28$

EXAMPLE 13

Bradley deposits a birthday gift of R14 000 into a savings account in order to save up for an overseas trip in six years' time. At the end of the second year, he withdraws R2 000 from the account. How much money will he have saved at the end of the six-year period, assuming that the interest rate for the whole savings period is 8% per annum compounded monthly?



Method 1

Value of 14 000 at T_2:

$$14\ 000\left(1+\frac{0,08}{12}\right)^{24}$$

$= 16420{,}43104$

Now subtract 2 000 from the savings account:

$16420{,}43104 - 2000$

$= 14420{,}43104$

Grow the balance to T_6:

$$14420{,}43104\left(1+\frac{0,08}{12}\right)^{48}$$

$= \text{R}19\ 837{,}70$

Method 2

$$A = \left[14000\left(1+\frac{0,08}{12}\right)^{24} - 2000\right]\left(1+\frac{0,08}{12}\right)^{48}$$

$\therefore A = \text{R}19\ 837{,}70$

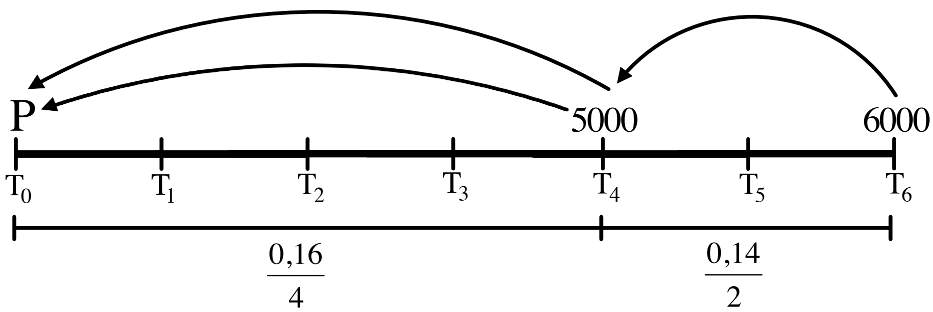
Method 3 **(Recommended)**

$$A = 14000\left(1+\frac{0,08}{12}\right)^{72} - 2000\left(1+\frac{0,08}{12}\right)^{48}$$

$\therefore A = R19\ 837,70$

EXAMPLE 14

Brian takes out a loan to finance the purchasing of a new DVD player. He repays the loan by means of a payment of R5 000 four years after the granting of the loan. Two years later, he repays a final amount of R6 000. The interest rate during the first four years of the loan is 16% per annum compounded quarterly. For the remaining two years, the interest rate changes to 14% per annum compounded half-yearly. How much money did Brian originally borrow?



Method 1

We will first determine the present value of R6 000 at T_4:

$$6000\left(1+\frac{0,14}{2}\right)^{-4}$$

Add the R5 000 to the present value:

$$6000\left(1+\frac{0,14}{2}\right)^{-4} + 5000$$

Now determine the present value at T_0:

$$P = \left[6000\left(1+\frac{0,14}{2}\right)^{-4} + 5000\right]\left(1+\frac{0,16}{4}\right)^{-16}$$

$\therefore P = R5113,44$

Method 2 (recommended)

Treat each amount separately and determine the present value of each amount at T_0. Then add the present values:

$$P = 6000\left(1+\frac{0,14}{2}\right)^{-4} . \left(1+\frac{0,16}{4}\right)^{-16} + 5000\left(1+\frac{0,16}{4}\right)^{-16}$$

$\therefore P = 2443,895945 \quad + \quad 2669,540878$

$\therefore P = R5113,44$

2443,895945 is the present value of 6 000 at T_0 (interest is removed from 6 000 to get the present value).
2669,540878 is the present value of 5 000 at T_0 (interest is removed from 5 000 to get the present value).
If you now add the interest free amounts, you get the value of the original loan.

Notice:
The two methods are equivalent to each other because of the distributive law:

$$\left[6000\left(1+\frac{0,14}{2}\right)^{-4}+5000\right]\left(1+\frac{0,16}{4}\right)^{-16}$$

$$=6000\left(1+\frac{0,14}{2}\right)^{-4}.\left(1+\frac{0,16}{4}\right)^{-16}+\ 5000\left(1+\frac{0,16}{4}\right)^{-16}$$

EXERCISE 6

1. Jennifer received a gift of R4 000 from her boyfriend. She deposited the money into a savings account. Three years later she received another gift of R5 000 from her boyfriend and deposited the money into the savings account. Calculate how much money Jennifer will have saved three years after her deposit of R5 000. Assume that the interest rate for the first two years of the savings period is 18% per annum compounded monthly and that it changes to 18% per annum compounded half-yearly for the remaining four years.
2. Mvelo decides to start an emergency savings account for the future. He deposits R6 000 into a savings account. Three years later, he deposits a further R8 000 into the account. Four years after this, he deposits a further R10 000 into the account. The interest rate for the first four years is 14% per annum compounded semi-annually. For the next three years the interest rate increases to 15% per annum compounded quarterly. Calculate the future value of the savings at the end of the seven-year period.
3. Jason wants to travel overseas in six years' time. He invests R15 000 in a savings account in order to save up for the overseas trip. The interest rate for the six-year period is 13% per annum compounded semi-annually. At the end of the fourth year he runs into financial difficulty and withdraws R3 000 from the account. How much money will he have saved at the end of the six-year period?
4. Mandy has just finished reading a book on the importance of saving for the future. She immediately opens a savings account and deposits R5 000 into the account. Two years later, she deposits a further R6 000 into the account. Thirty six months later, she withdraws R3 000 to buy a birthday gift for her husband. The interest rate during the first three years of the investment is 8% per annum compounded monthly. The interest rate then changes to 9% per annum compounded quarterly. Calculate the value of Mandy's investment two years after her withdrawal of R3 000.

5. Leandra deposits R4300 into a savings account. Three years later, she deposits a further R7 000 into the account. Two years later, she withdraws R2 000. Two years after this, she deposits a final amount of R1 000 into the account. The interest rate during the first three years is 13% per annum compounded monthly. For the remaining four years, the interest rate changes to 14% per annum compounded quarterly. Calculate the future value of the savings at the end of the seven-year period.
6. Bob is excited about the fact that he has managed to save R1 000 000 after ten years. The interest rate during the first four years was 18% per annum compounded monthly. Thereafter and for the remaining six years, the interest rate changed to 20% per annum compounded quarterly. Calculate Bob's original amount at the start of the savings plan.
7. Stephanie repaid a seven-year loan by means of R2300 four years after the granting of the loan and R4200 three years thereafter. The interest rate for the first five years was 10% per annum compounded half-yearly and 12% per annum compounded monthly for the remaining period. What was the amount she originally borrowed?
8. Arnold buys a car and pays an initial deposit of R6 000, R8 000 after two years and a further R9 000 two years later. Interest is 18% per annum compounded monthly for the first two years. For the next two years, the interest rate is 19% per annum compounded monthly. Calculate the original price of the car.
9. Sipho pays a deposit of R30 000 for a new car. He repays the remaining loan by paying a further R20 000 in two years' time and R100 000 six years thereafter. Interest is 18% per annum compounded monthly during the first three years and 32% per annum effective for the remaining five years. What did Sipho originally pay for the car?
10. A mother decided to start saving money for her son's future education. She immediately deposited R4 000 into a savings account. Three years later, she deposited a further R5 000 into the account. One year later, she withdrew R2 000 in order to do repairs around the house. Her son needed the money four years after her withdrawal of R2 000. The interest rate for the first three years was 15% per annum compound monthly. The interest rate for the remaining five years was 16% per annum compounded quarterly. Calculate the future value of his money at the end of the savings period which lasted eight years.
11. Jacob borrowed money in order to finance the purchase of a new home. He paid a deposit of R100 000. He then repaid the balance that he owed by means of a payment of R160 000 after three years. His last payment of R170 000 was paid four years after the previous payment. The interest rate for the first three years was 12% per annum compounded monthly. The interest rate was then 14% per annum compounded half-yearly for the remaining four years. What was the original price of the home?

REVISION EXERCISE

1. Lindiwe wants to save up for a 20% deposit on a car she intends buying in two years' time. She invests R27 000 at an interest rate of 9% per annum compounded monthly.
   (a) If the car will cost R160 000 in two years' time, will she have saved enough money to pay the 20% deposit?
   (b) In two years' time, Lindiwe pays the deposit and buys the car. She arranges to pay the car in full (with interest) with a once-off payment in three years' time. Calculate the amount of this payment if the interest charged is 15% per annum compounded monthly.
   (c) Lindiwe's car will depreciate at 6% per annum on a reducing balance scale. Calculate the depreciated value (trade-in value) of her car three years after she bought it.
   (d) Lindiwe has paid off her old car in three years and decides to trade in the car and buy a new one. She agrees to make two future payments in order to pay for the new car. The first payment of R80 000 will be made in one years' time and the second payment of R100 000 will be made two years after her first payment. The interest rate is 18% per annum compounded quarterly. If she uses the money received from the trade-in of her old car, how much will the new car cost?
2. Michael invests R10 000 for a period of six months at an interest rate of 8% per annum compounded monthly.
   (a) Calculate the amount saved after six months using the nominal rate.
   (b) Convert the nominal rate to the annual effective rate.
   (c) Calculate the amount saved using the annual effective rate.
   (d) If the interest rate was changed to 8% **per month effective**, calculate the amount saved after six months.
   (e) If the interest rate was changed to 8% **per quarter effective**, calculate the amount saved after six months.
3. On the 1^{st} January 2005, Diana deposited R15000 into a savings account. On the 1^{st} January 2007, she deposited double her first amount into the account. On the 1^{st} January 2008, she deposited double her second amount into the account. The interest rate for the first two years (2005-2006) was 9% per annum compounded monthly. The interest rate then changed to 10% per annum compounded monthly. Calculate the value of her investment on the 1^{st} January 2009.
4. Michael deposits a gift of R20 000 into a savings account in order to save up for an overseas trip in six years' time. The interest rate for the savings period is 8% per annum compounded monthly for the first two years and 9% per annum compounded quarterly for the remaining four years.
   (a) How much money will he have saved at the end of the six-year period?
   (b) Suppose that at the end of the second year, he withdraws R4 000 from the account. How much money will he have then saved at the end of the six year period?
   (c) Instead of withdrawing money, suppose that at the end of the third year, he adds R3 000 to the savings. How much money will he have then saved at the end of the six-year period?

5. John takes out a loan to finance the purchasing of a new computer laptop. He repays the loan by means of a payment of R6 000 four years after the granting of the loan. Two years later, he repays a final amount of R8 000. The interest rate during the first four years of the loan is 12% per annum compounded quarterly. For the remaining two years, the interest rate changes to 16% per annum compounded half-yearly. How much money did John originally borrow?

SOME CHALLENGES

1. How much (correct to the nearest rand) do I need to invest now at 9% per annum compounded quarterly, to provide for the purchase of two airline tickets in two years' time and another two in four years' time? The first two tickets are expected to cost R18 000 (R9 000 per ticket). An inflation rate of 6% per annum must be taken into consideration in the two years between the purchase of the first and second set of tickets.
2. Simphiwe deposits Rx into a unit trust savings account. Two years later, she deposits a further R2x into the account. Three years after this, she deposits R3x into the account. The interest rate for this five year period is 18% per annum compounded monthly. She receives R60 000 at the end of the five year period. Calculate the value of x.
3. Sean opens a savings account and deposits R5 000 into the account immediately. Five months later he invests a further Rx into the account. Three months later, he deposits R2x into the account. The interest rate for the first four months is 18% per annum compounded monthly, 21% per annum compounded monthly for the next two months and 24% per annum compounded monthly for the remaining two months. The investment has a future value of R100 000 at the end of the 8^{th} month. Calculate the value of the final payment at the end of the 8^{th} month.
4. An amount is invested in an account which pays interest at a rate of 16% per annum compounded monthly. After five years, half the amount in the account at that stage is withdrawn and deposited into a new account which pays 18% per annum effective. The other half of the accumulated amount is left in the original account. At the end of a further five years, the combined total in the two accounts is R29 896,78. How much was originally deposited in the first account?

CHAPTER 11 – STATISTICS

Before studying the content of this chapter, it is extremely important to revise the concepts dealt with in the Grade 10 book. Determining the mean, median, mode, lower and upper quartiles, range, inter-quartile range and semi-interquartile range will be required in this module. You also need to revise frequency diagrams, stem and leaf plots, bar graphs, histograms, pie charts and box and whisker plots.

EXERCISE 1 **(REVISION OF GRADE 10 STATISTICS)**

1. The results for Argentina for the past 14 World Cup tournaments are recorded in the table below.

WC Tournament	Matches played	Wins	Draws	Losses	Goals for	Goals against
2006 Germany	5	3	2	0	11	3
2002 Japan	3	1	1	1	2	2
1998 France	5	3	1	1	10	4
1994 USA	4	2	0	2	8	6
1990 Italy	7	2	3	2	5	4
1986 Mexico	7	6	1	0	14	5
1982 Spain	5	2	0	3	8	7
1978 Argentina	7	5	1	1	15	4
1974 Germany	6	1	2	3	9	11
1966 England	4	2	1	1	4	2
1962 Chile	3	1	1	1	2	3
1958 Sweden	3	1	0	2	5	10
1934 Italy	1	0	0	1	2	3
1930 Uruguay	5	4	0	1	18	9

Source: www.2010 Fifa World Cup:Statistics – MediaClubSouthAfica.com

(a) Calculate the mean for the number of matches played over the 14 tournaments.
(b) Calculate the mean for the number of goals scored for Argentina played over the 14 tournaments.
(c) What is the mode for the number of matches played?
(d) What is the mode for the number of losses?
(e) Determine the quartiles for the number of matches played.
(f) Determine the quartiles for the number of goals scored for Argentina.
(g) Calculate the interquartile range for the number of matches played.
(h) Calculate the interquartile range for the number of goals scored for Argentina.
(i) What is the mode for the goals scored for Argentina?
(j) What is the mode for the games won by Argentina?
(k) What is the mode for the goals scored against Argentina?
(l) What is the mode for the games in which Argentina scored a draw?
(m) Draw box and whisker plots for the matches played by Argentina, the wins and the goals scored against Argentina.

(n) Comment on the performance of Argentina over the 14 tournaments (refer to 1(m)).

2. The ages of the final 23 players selected by coach Oscar Tabarez to play for Uruguay in the 2010 FIFA World Cup are provided below.

Position	Player	Age
1	Fernando Musiera	23
2	Diego Lugano (captain)	29
3	Diego Godin	24
4	Jorge Fucile	25
5	Walter Gargano	25
6	Andres Scotti	35
7	Edinson Cavani	23
8	Sebastian Eguren	29
9	Luis Suarez	23
10	Diego Forlan	31
11	Alvaro Perreira	25
12	Juan Castillo	32
13	Sebastian Abreu	33
14	Nicolas Lodeira	23
15	Diago Perez	30
16	Maxi Perrreira	26
17	Ignacio Gonzales	28
18	Egidio Arevalo Rios	27
19	Sebastian Fernandes	25
20	Mauricio Victorino	27
21	Alvaro Fernandez	24
22	Martin Caceres	23
23	Martin Silva	27

Source: www.2010 Fifa World Cup: squads MediaClubSouthAfica.com

The ages of the players are to be grouped into class intervals.

(a) Redraw and complete the following table:

Class intervals (ages)	Frequency	Midpoint of class interval
$16 \leq x < 20$		
$20 \leq x < 24$		
$24 \leq x < 28$		
$28 \leq x < 32$		
$32 \leq x < 36$		
$36 \leq x < 40$		

(b) Draw a histogram and frequency polygon for this data. (A grid is provided in the Teacher’s Guide – this may be photocopied for the learners)

(c) Calculate the actual mean age of the Uruguay team.

(d) Using the table, calculate the estimated mean age for this team.

(e) Write down the modal age.

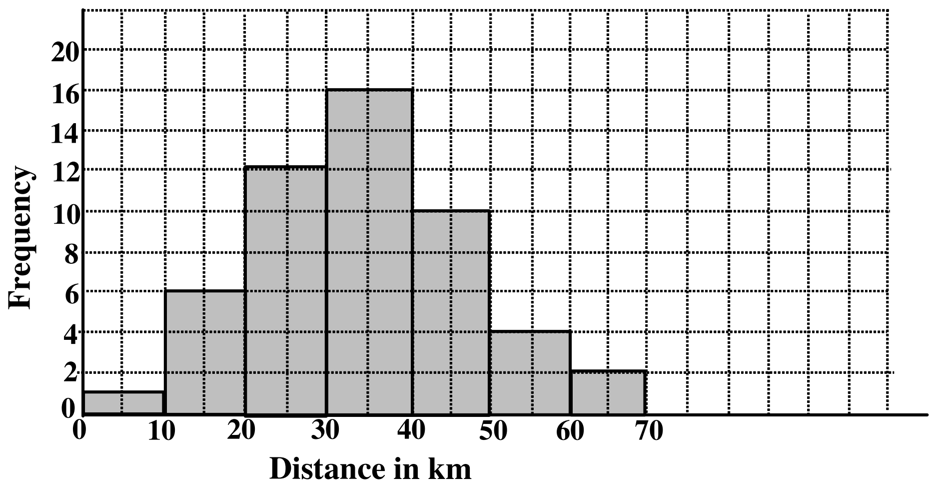
(f) Write down the median age.
(g) Write down the range.
(h) Write down the lower and upper quartile ages for the data.
(i) Write down the semi-interquartile range.

3. The annual earnings (in pounds) of the top 20 soccer players during 2011 are represented as grouped data in the following table.

Class intervals (in millions of pounds)	Frequency (number of players)
$5 \le x < 10$	9
$10 \le x < 15$	5
$15 \le x < 20$	2
$20 \le x < 25$	1
$25 \le x < 30$	3

source: www.zimbrio.com/Football's-richest-players-and-managers
(a) Calculate the estimated mean for this data.
(b) Calculate the estimated median for this data.
(c) Write down the modal class interval for this data.

4. The following histogram represents the distance run by marathon runners.



(a) Redraw and complete the following table:

Class interval	Frequency
$0 < x \le 10$	
$10 < x \le 20$	
$20 < x \le 30$	
$30 < x \le 40$	
$40 < x \le 50$	
$50 < x \le 60$	
$60 \le t \le 70$	

(b) Calculate the estimated mean.

CUMULATIVE FREQUENCY GRAPHS

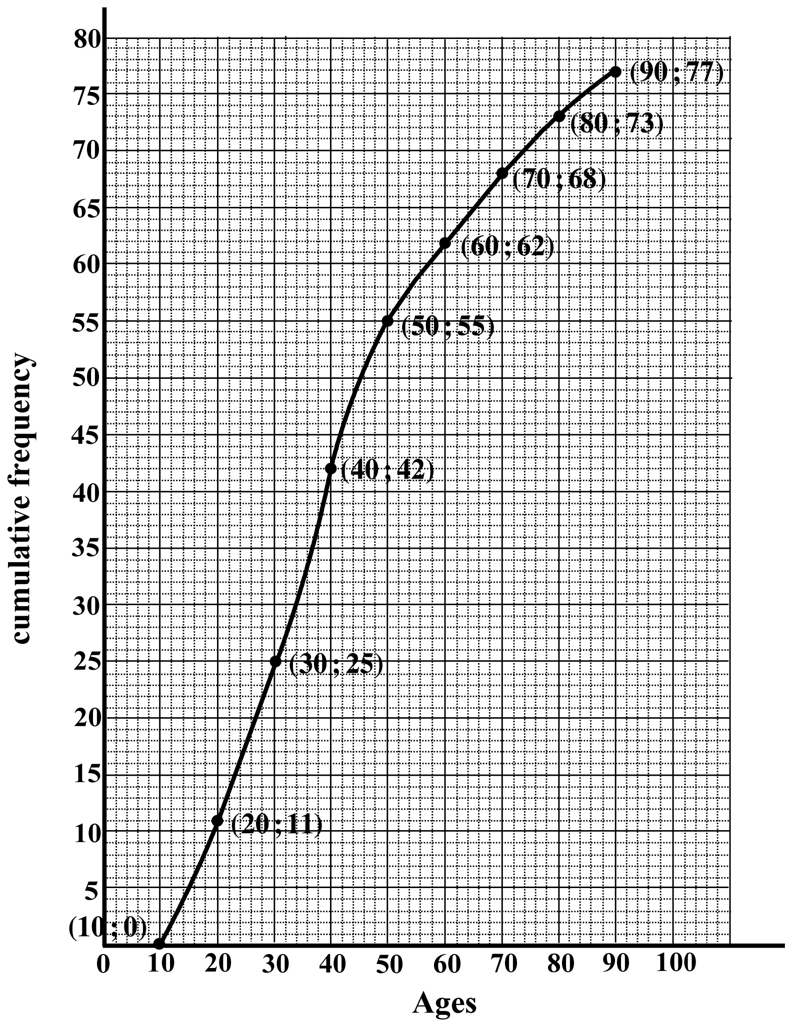
Determining **cumulative frequencies** is an effective way of representing **grouped data**. If you want to find the median of grouped data from a frequency table, a useful way to do this is by first determining the cumulative frequencies from the frequency table and then representing the information on a **cumulative frequency graph (or ogive curve)**. The method is illustrated below.

EXAMPLE 1

The company HEALTHMANIA conducted a survey in Gauteng to find out which age group most frequently uses their health supplements. The company determined the ages of a representative sample of their current client group. The ages of current clients were recorded and then sorted. The company wanted to market a new health supplement to the age group in Gauteng which most frequently uses their products.

Class interval	Frequency	Cum Freq		Graph points
$0 \leq x \leq 10$	0	0		
$10 < x \leq 20$	11	$0+11=11$	11 people were 20 years or less	$(20\,;11)$
$20 < x \leq 30$	14	$11+14=25$	25 people were 30 years or less	$(30\,;25)$
$30 < x \leq 40$	17	$25+17=42$	42 people were 40 years or less	$(40\,;42)$
$40 < x \leq 50$	13	$42+13=55$	55 people were 50 years or less	$(50\,;55)$
$50 < x \leq 60$	7	$55+7=62$	62 people were 60 years or less	$(60\,;62)$
$60 < x \leq 70$	6	$62+6=68$	68 people were 70 years or less	$(70\,;68)$
$70 < x \leq 80$	5	$68+5=73$	73 people were 80 years or less	$(80\,;73)$
$80 < x \leq 90$	4	$73+4=77$	77 people were 90 years or less	$(90\,;77)$
$90 < x \leq 100$	0			
	Total: 77			

Notice:
The total frequency of marks (77) is equal to the final cumulative frequency (77).
We can now represent the information graphically. The graph below is called a **cumulative frequency graph** or **ogive curve**.
The horizontal axis represents the age class intervals. The vertical axis represents the cumulative frequencies of the ages (0 –77).



Important Deductions

1. As the number of points increases, the graph will take on the form of an S-shaped curve.

2. For each point, the **first coordinate** represents the **upper boundary** of the class interval. The **second coordinate** represents the **cumulative frequency** of the ages. For example, for the point $(20\,;11)$, there were 11 people whose ages were less than 20 years. The upper boundary here is 20.

3. We can use the graph to determine estimates of the quartiles for this data.

   The **median** is to be found at the 39^{th} client (approximately half way up). 38 clients are below the 39^{th} client and 38 clients are above the 39^{th} client. $(38 \text{ clients} + 39\text{th client} + 38 \text{ clients} = 77 \text{ clients})$.
   You can read off the approximate median age by using the graph. Draw a horizontal line from 39 on the vertical axis. Then draw a vertical line down to the *x*-axis and determine an approximate value for the median, which is an age of 38.
   Alternatively, you can use the formula $\frac{1}{2}(n+1)$ to determine the position of the median (see page 288 of Teacher guide for notes on this):
   Position of median $= \frac{1}{2}(77+1) = 39\text{th position}$

The **lower quartile** is the median of the lower half of the data set. The lower half contains 38 values, which is even. Therefore, the lower quartile must be inserted between the 19^{th} and 20^{th} client (approximately a quarter of the way up).

19 clients are below the inserted lower quartile and 19 clients are above the inserted lower quartile.

To get the lower quartile, draw a horizontal line from 19,5 on the y-axis to the graph. Then draw a vertical line down to the x-axis. Read off the approximate lower quartile age, which is approximately 26.

Alternatively, you can use the formula $\frac{1}{4}(n+1)$ to determine the position of the lower quartile:

Position of lower quartile $= \frac{1}{4}(77+1) = 19{,}5$th position

The **upper quartile** is the median of the upper half of the data set. The upper half contains 38 values, which is even. This means that the upper quartile will be an inserted number which is the average between the two upper half numbers. There are 19 clients below this inserted upper quartile and 19 clients above this inserted upper quartile.

Working from the 39^{th} client (median), simply add 19 to get you to the 58^{th} client. Therefore, the upper quartile must be inserted between the 58^{th} and 59^{th} client (approximately three quarters of the way up).

To get the upper quartile, draw a horizontal line from 58,5 on the y-axis to the graph. Then draw a vertical line down to the x-axis. Read off the approximate upper quartile age, which is approximately 54.

Alternatively, you can use the formula $\frac{3}{4}(n+1)$ to determine the position of the upper quartile:

Position of upper quartile $= \frac{3}{4}(77+1) = 58{,}5$th position

Note:

If the decimal for the position of the quartile works out to be 0,25 or 0,75, then round down in the case of 0,25 and round up in the case of 0,75. You are simply determining estimates of the quartiles rather than the exact values. Alternatively, don't round up or down and use the decimals in your readings.

For example, if the position of the quartile is 19,75 you can either round up to 20, draw a horizontal line from the y-axis to the graph and then draw a line down to the x-axis and estimate the quartile. Alternatively, draw the horizontal line from approximately 19,75 on the y-axis.

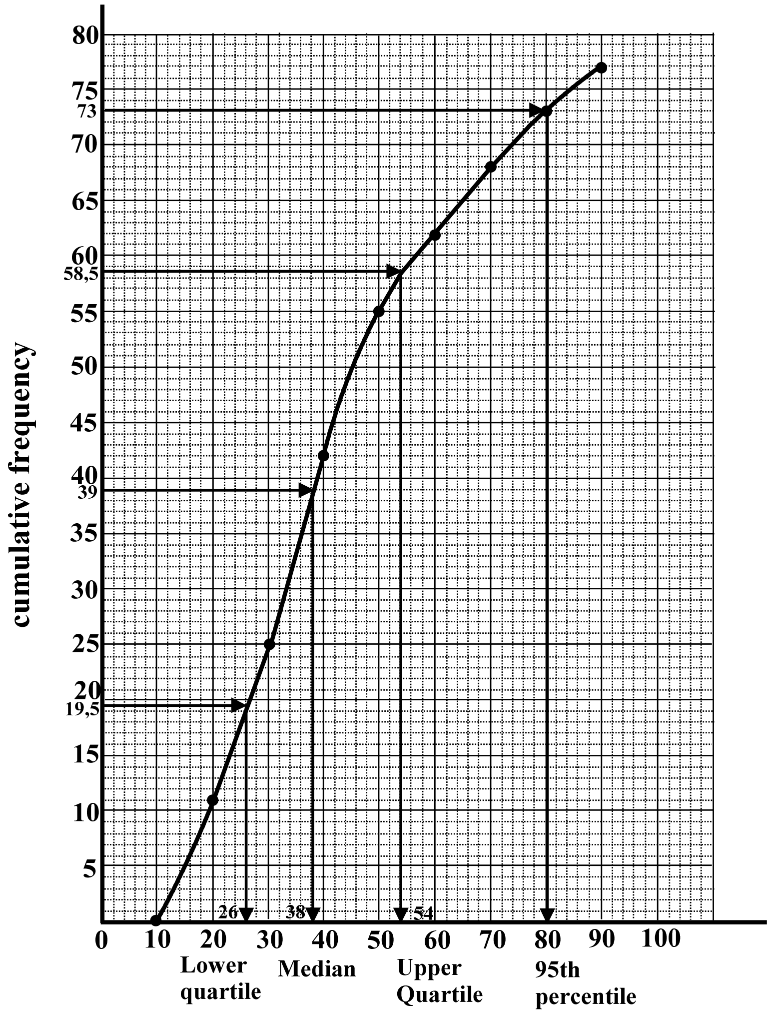
For extension notes on determining the position of quartiles using these three formulae, refer to the Teacher's Guide page 288.

4. We can determine **percentiles** using the graph. Percentiles divide the data into hundredths.

For example, the 95^{th} percentile can be determined as follows:

$$\frac{95}{100} \times 77 = 73{,}15 \approx 73\text{rd client} \ \text{(rounded off to the nearest whole number)}$$

To get the 95^{th} percentile, draw a horizontal line from 73 on the y-axis to the graph. Then draw a vertical line down to the x-axis. Read off the approximate upper quartile age, which is approximately 80.



EXERCISE 2 (Photocopyable grids are provided in the Teacher's Guide)

1. The following table shows the marks obtained by 220 learners in a Mathematics examination.

Percentage	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	2	6	11	22	39	59	45	20	11	5

(a) Redraw and complete the cumulative frequency for this data.

Marks	Frequency	Cumulative frequency
$1 \leq x \leq 10$		
$11 \leq x \leq 20$		
$21 \leq x \leq 30$		
$31 \leq x \leq 40$		
$41 \leq x \leq 50$		
$51 \leq x \leq 60$		
$61 \leq x \leq 70$		
$71 \leq x \leq 80$		
$81 \leq x \leq 90$		
$91 \leq x \leq 100$		
Total		

(b) Draw a cumulative frequency graph (ogive curve) for this data.
(c) Determine the quartiles.
(d) Determine the 90^{th} percentile.
(e) Determine the 10^{th} percentile.

2. The following table represents the percentage of monthly income spent on petrol and car expenses by fifty people.
(a) Redraw and complete the cumulative frequency for this data.

Percentage	**Frequency**	**Cumulative frequency**
$12 \leq p < 18$	8	
$18 \leq p < 24$	20	
$24 \leq p < 30$	12	
$30 \leq p < 36$	8	
$36 \leq p \leq 42$	2	
Total		

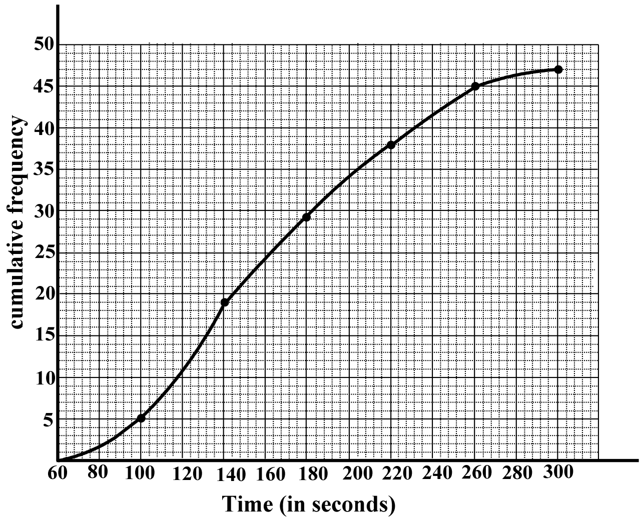
(b) Draw a cumulative frequency graph (ogive curve) for this data.
(c) Determine the quartiles.
(d) Determine the 25^{th}, 50^{th} and 75^{th} percentiles. What do you notice?
(e) Determine the 80^{th} percentile.

3. Fifty motorists were asked to record the number of kilometres travelled in one week. The following table shows the results:

Number of kilometres	**Number of motorists**	**Cumulative frequency**
$10 < x \leq 20$	2	2
$20 < x \leq 30$		9
$30 < x \leq 40$		13
$40 < x \leq 50$		26
$50 < x \leq 60$		42
$60 < x \leq 70$		50

(a) Redraw and complete the table.
(b) Draw the cumulative frequency curve for the data.
(c) Use your graph to estimate the median number of kilometres per week.

4. The following cumulative frequency graph shows the times between planes landing at an airport.



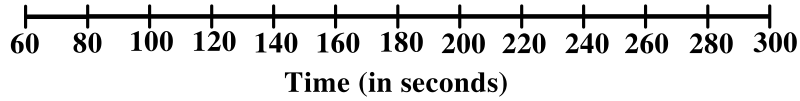
(a) Redraw and complete the following table:

Times between planes (seconds)	Frequency	Cumulative frequency
$60 \leq t < 100$		5
$100 \leq t < 140$		
$140 \leq t < 180$		
$180 \leq t < 220$		
$220 \leq t < 260$		
$260 \leq t \leq 300$		

(b) Determine the estimated median time between planes landing at the airport.

(c) How many planes had a time between them of 220 seconds or more?

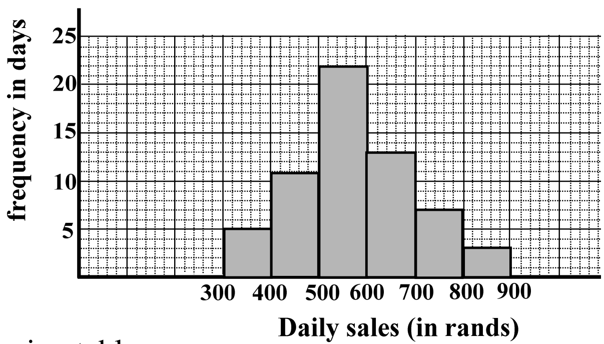
(d) Redraw the number line below and draw a box and whisker diagram for the times between planes.



(e) Calculate an estimate of the mean for the time between planes. Redraw and use the following table to assist you. Round off your answer to the nearest whole number.

Time between planes	Frequency		
$60 \leq t < 100$			
$100 \leq t < 140$			
$140 \leq t < 180$			
$180 \leq t < 220$			
$220 \leq t < 260$			
$260 \leq t < 300$			

5. A small coffee shop has kept a record of sales for the past two months. The daily sales, in rands (not in cents), is shown in the following histogram.



(a) Redraw and complete the following table:

Class interval	Frequency	Cumulative frequency
$300 < t \leq 400$		5
$400 < t \leq 500$		
$500 < t \leq 600$		
$600 < t \leq 700$		
$700 < t \leq 800$		
$800 < t \leq 900$		

(b) Draw an ogive curve for the sales over the past two months.

(c) Determine the estimated median value for the daily sales.

(d) Estimate the interval of the upper 25% of the daily sales.

VARIANCE AND STANDARD DEVIATION (UNGROUPED DATA)

Standard deviation (s or σ) is a measure of how spread-out numbers are. It is the square root of the **variance** (s^2), which is the average of the squared differences from the mean. Unlike the range that only considers the extreme values, the variance considers all the data points and then determines their distribution.

Before carrying on, we need to introduce some notation for the mean.
The mean of a set of data is defined as $\overline{x}$ (x bar).

$$\overline{x} = \frac{\text{sum of the values}}{\text{number of the values}} = \frac{\sum x}{n}$$

EXAMPLE 2

Calculate the mean of the following data:

13, 14, 14, 16, 17, 17, 17, 17, 17, 18, 18, 18, 19, 19, 19, 20

$$\overline{x} = \frac{\sum x}{n}$$

$$= \frac{13+14+14+16+17+17+17+17+17+18+18+18+19+19+19+20}{16}$$

$$= \frac{273}{16} = 17{,}1$$

EXAMPLE 3 (without using a calculator)

A chess team consisting of 10 players scored the following points during the year:

23 34 39 40 42 53 56 62 68 76

(a) Calculate the mean rounded off to one decimal place.
(b) Calculate the variance rounded off to two decimal places.
(c) Calculate the standard deviation rounded off to one decimal place.
(d) Determine the **standard deviation intervals** for the data.
(e) Make **conclusions** about the spread of the data about the mean by establishing how many of the data values lie within or outside of the first standard deviation interval.

Solutions

(a) $\overline{x} = \frac{\sum x}{n} = \frac{23+34+39+40+42+53+56+62+68+76}{10} = \frac{493}{10} = 49{,}3$

(b) Calculate the **individual deviations from the mean**. Record your results in a table.

POINTS SCORED (x)	$(x-\overline{x})$
23	$23-49{,}3=-26{,}3$
34	$34-49{,}3=-15{,}3$
39	$39-49{,}3=-10{,}3$
40	$40-49{,}3=-9{,}3$
42	$42-49{,}3=-7{,}3$
53	$53-49{,}3=3{,}7$
56	$56-49{,}3=6{,}7$
62	$62-49{,}3=12{,}7$
68	$68-49{,}3=18{,}7$
76	$76-49{,}3=26{,}7$
	$\sum(x-\overline{x})=0$

You will probably notice that the sum of all these deviations from the mean equals 0, which is not very helpful if we are to make conclusions about how the data is spread about the mean.

Calculate the **individual squared deviations from the mean**. Record your results in a table. Add these squared deviations. By squaring the deviations, we are able to eliminate the negative signs so that we have more useful values.

POINTS SCORED (x)	$(x-\overline{x})$	$(x-\overline{x})^2$
23	$23-49{,}3=-26{,}3$	691,69
34	$34-49{,}3=-15{,}3$	234,09
39	$39-49{,}3=-10{,}3$	106,09
40	$40-49{,}3=-9{,}3$	86,49
42	$42-49{,}3=-7{,}3$	53,29
53	$53-49{,}3=3{,}7$	13,69
56	$56-49{,}3=6{,}7$	44,89
62	$62-49{,}3=12{,}7$	161,29
68	$68-49{,}3=18{,}7$	349,69
76	$76-49{,}3=26{,}7$	712,89
	$\sum(x-\overline{x})=0$	$\sum(x-\overline{x})^2=2454{,}1$

Determine the **variance** (mean of the squared deviations) by using the following formula: $s^2=\dfrac{\sum(x-\overline{x})^2}{n}$

$$\therefore s^2=\frac{\sum(x-\overline{x})^2}{n}=\frac{2454{,}1}{10}=245{,}41$$

(c) Determine the **standard deviation (*s*)** by square rooting the variance.

Standard deviation: $s = \sqrt{\dfrac{\sum (x-\overline{x})^2}{n}} = \sqrt{245,41} = 15,7$

(d) One standard deviation interval: $(\overline{x} - s\,;\overline{x} + s)$
Two standard deviation interval: $(\overline{x} - 2s\,;\overline{x} + 2s)$
Three standard deviation interval: $(\overline{x} - 3s\,;\overline{x} + 3s)$
For the example used, the standard deviation intervals are:
One standard deviation interval:
$(49,3-15,7\,;49,3+15,7) = (33,6\,;65)$
Two standard deviation intervals:
$(49,3-31,4\,;49,3+31,4) = (17,9\,;80,7)$
Three standard deviation intervals:
$(49,3-47,1\,;49,3+47,1) = (17,9\,;80,7)$

(e) In this example, it is clear that 7 of the 10 points lie within the one standard deviation interval (34, 39, 40, 42, 53, 56 and 62). This means that 70% of the points lie within the one standard deviation interval. It is therefore evident that most of the players performed well by scoring close to the mean points score. The teamwork was very good for this team. All points lie within the two standard deviation interval.
The standard deviation intervals are useful when the data set is reasonably large. You can even work out the three standard deviation interval with large data.

<u>EXAMPLE 3</u> **(by using a calculator)**

The standard deviation can be easily calculated by using a calculator. Refer to your calculator's manual for the details on how to do this or discuss this with your teacher.

<u>EXAMPLE 4</u> **(without using a calculator)**

Consider the table which shows the number of learners who obtained certain marks out of 30 for a class test.

Marks (*x*)	18	19	20	21	22	23	24	25	26	27
No of learners	2	2	3	4	6	9	12	6	5	3

Calculate the mean and standard deviation for this data (one decimal place).

<u>Solution</u>

Marks x	**Freq** f	$f \times x$	$x-\overline{x}$	$(x-\overline{x})^2$	$f \times (x-\overline{x})^2$
18	2	36	−5,2	27,04	54,08
19	2	38	−4,2	17,64	35,28
20	3	60	−3,2	10,24	30,72

21	4	84	–2,2	4,84	19,36
22	6	132	–1,2	1,44	8,64
23	9	207	–0,2	0,04	0,36
24	12	288	0,8	0,64	7,68
25	6	150	1,8	3,24	19,44
26	5	130	2,8	7,84	39,2
27	3	81	3,8	14,44	43,32
Total	**52**	$\overline{x} = \frac{1206}{52} = 23{,}2$			258,08

The mean for this data is 23,2.

$$s^2 = \frac{\sum f(x-\overline{x})^2}{n} = \frac{258{,}08}{52} = 4{,}963076923 \qquad \text{(variance)}$$

The standard deviation is:

$$s = \sqrt{\frac{\sum f(x-\overline{x})^2}{n}} = \sqrt{4{,}963076923} = 2{,}2$$

EXAMPLE 4 **(by using a calculator)**

The standard deviation can be easily calculated by using a calculator. Refer to your calculator's manual for the details on how to do this or discuss this with your teacher.

EXERCISE 3

1. The number of points scored by four cyclists over a number of races are given below:

A	1	1	1	2	6	6	8	8	8	8	10	10	10
B	1	2	6	8	8	8	8	8	8	10	10	10	-
C	1	1	2	2	4	4	6	6	8	8	10	-	-
D	2	2	2	4	4	6	6	8	8	10	10	10	-

(a) Calculate the mean and standard deviation for each cyclist without using a calculator.
(b) Now calculate the standard deviation for each cyclist by using your calculator.
(c) Discuss the performance of each cyclist by referring to the standard deviation for each cyclist.

2. The following table contains the number of learners who obtained certain marks on a Biology class test out of 30.

Marks	20	21	22	23	24	25	26	27	28	29
No of learners	3	3	4	5	7	10	13	5	4	2

(a) Calculate the standard deviation for this data by means of drawing a frequency table.
(b) Now calculate the standard deviation using your calculator.

3. The maximum daily temperatures for Pretoria for the first ten days in September are recorded in the following table:

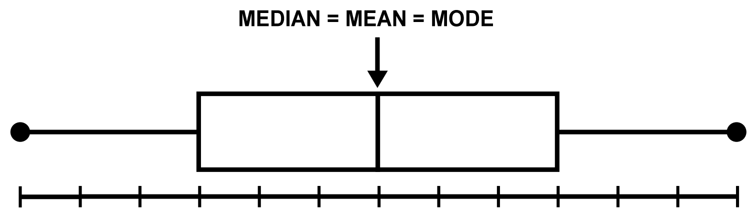
18	17	24	28	27	20	23	25	22	25

(a) Calculate the standard deviation.
(b) Calculate the variance rounded off to one decimal place.

4. Five data values are represented as follows:
$2x$; $x+1$; $x+2$; $x-3$; $2x-2$
(a) Determine the value of x if the mean of the data set is 15.
(b) Draw a box and whisker plot for the data values.
(c) Calculate the inter-quartile range.
(d) Calculate the standard deviation for this data, rounded off to one decimal place.
(e) Calculate the variance rounded off to one decimal place.

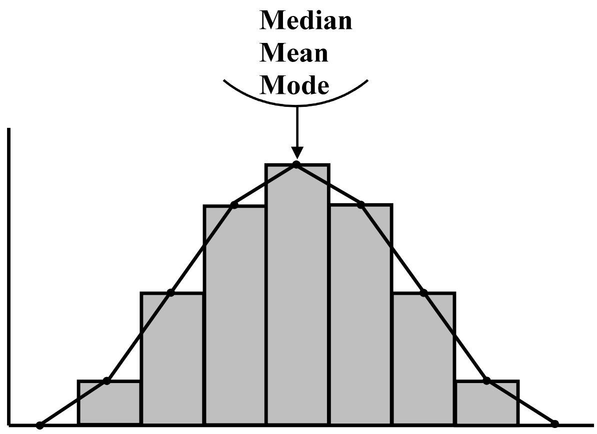
DISTRIBUTION OF DATA

NORMALLY DISTRIBUTED OR SYMMETRICAL DATA



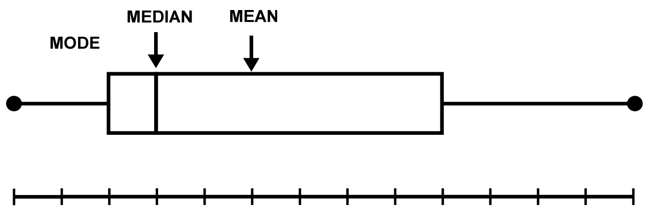
If the data to the left of the median balances with the data on the right, then the mean, median and mode will be the same. The data is said to be **normally distributed or symmetrical.**

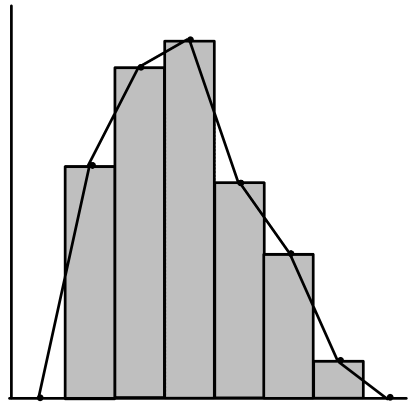
Mean = Median = Mode



DISTRIBUTIONS THAT ARE NOT NORMALLY DISTRIBUTED

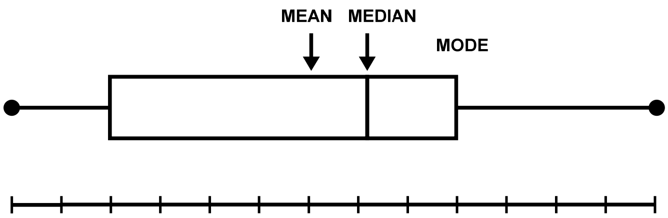
Situation 1

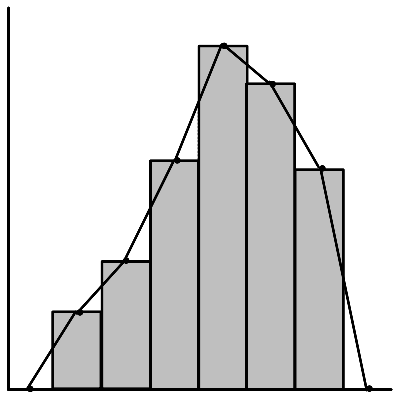




If there is a mass of data values predominantly on the left side of the distribution with fewer higher values on the right, the data is said to be **positively skewed or skewed to the right**. The mode will be closer to the left of the distribution. The median is to the left.
In this case: **Mean > Median**

Situation 2





If there is a mass of data values predominantly on the right side of the distribution with fewer lower values on the left, the data is said to be **negatively skewed or skewed to the left.** The mode will be closer to the right of the distribution. The median is to the right.
In this case: **Mean < Median**

EXERCISE 4

1. The results of four learners in a series of formative tests each out of 10 marks are recorded in the following table:

A	1	1	1	2	6	6	7	8	8	8	10	10	10
B	1	2	6	8	8	8	8	8	8	10	10	10	-
C	1	1	2	2	4	4	6	6	8	8	10	-	-
D	2	2	2	4	4	6	6	8	8	10	10	10	-

(a) Calculate the mean for each of the learners.
(b) List the Five Number Summary for each learner.
(c) Draw a Box and Whisker plot for each learner.

(d) Discuss each learner's distribution of scores in terms of the spread about the median and mean.

(e) Compare the performance results for each learner by using the information obtained above.

2. The Mathematics Paper 1 June results for Mr Mogodi's class of learners are recorded below. The exam was out of 150.

101	90	85	97	89	85	84	88
83	96	93	81	88	92	88	96
92	77	85	91	81	80	87	88

(a) Calculate the mean, median and mode for this data. What do you notice?

(b) Determine the lower and upper quartile.

(c) Calculate the standard deviation for this data.

(d) Determine the percentage of marks that lie within one, two and three standard deviations of the mean.

(e) Draw a histogram and frequency polygon to illustrate the above data. Indicate the position of the mean, median and mode on your diagram. Use class intervals of $70 \leq x < 74$, $74 \leq x < 78$, ……

(f) Explain why the data is normally distributed.

(g) Draw a box and whisker diagram to illustrate the normally distributed data.

3. The Science marks (out of 40) of Mrs Basson's learners are recorded below:

30	24	21	18	31	28	21	20	18
27	19	23	21	17	25	22	19	27
35	18	22	27	30	20	27	21	23

(a) Create a histogram and frequency polygon for the above data. Use class intervals of $12 \leq x < 16, 16 \leq x < 20$, $20 \leq x < 24$, $24 \leq x < 28$, ...

(b) Describe the shape of the frequency polygon and then predict the relationship between the mean and median.

(c) Now calculate the mean and the median and determine whether your prediction is correct.

(d) Draw a box and whisker diagram for the data in order to verify how the data is positively skewed.

OUTLIERS

In a set of data, it sometimes happens that a particular number is extremely high or low in comparison to the other numbers. Such a number is called an **outlier**.
In 1977, the statistician, John Tukey, invented box and whisker plots and defined an outlier to be any number in a data set which falls outside the interval:
$\left[Q_1 - \frac{3}{2} \times \text{IQR} \; ; Q_3 + \frac{3}{2} \times \text{IQR}\right]$ where Q_1 and Q_3 represent the lower and upper quartiles respectively. IQR represents the inter-quartile range ($\text{IQR} = Q_3 - Q_1$)

$Q_1 - \frac{3}{2} \times \text{IQR}$ is called the **lower fence** and $Q_3 + \frac{3}{2} \times \text{IQR}$ is called the **upper fence**.

Outliers are important to identify since they can distort the mean and have an adverse effect on the standard deviation. For example, if the annual earnings of Bill Gates is included with the annual earnings of nine school teachers, the average salary of the ten people would be completely distorted and way too high. Bill Gates earns in excess of 500 million US dollars per annum as compared to the low salaries earned by the teachers. His salary is an outlier and cannot be included in the data set when calculating the mean.

<u>EXAMPLE 5</u>

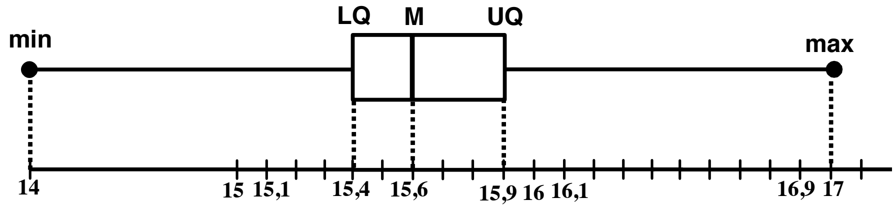
Consider the following set of numbers:

14	15,1	15,4	15,4	15,4	15,5	15,5	15,6
15,7	15,7	15,7	15,9	16,1	16,9	17,0	

(a) Draw a box and whisker plot for this data.
(b) Determine the outliers.
(c) Draw a box and whisker plot highlighting the outliers.
(d) Discuss the effect of the outliers on the mean, median and standard deviation if the outliers are included in the data set and if they are excluded.

<u>Solutions</u>

(a) $\text{Min} = 14$ $\quad Q_1 = 15{,}4$ $\quad M = 15{,}6$ $\quad Q_3 = 15{,}9$ $\quad \text{Max} = 17$



(b) Inter-quartile range (IQR) $= 15{,}9 - 15{,}4 = 0{,}5$

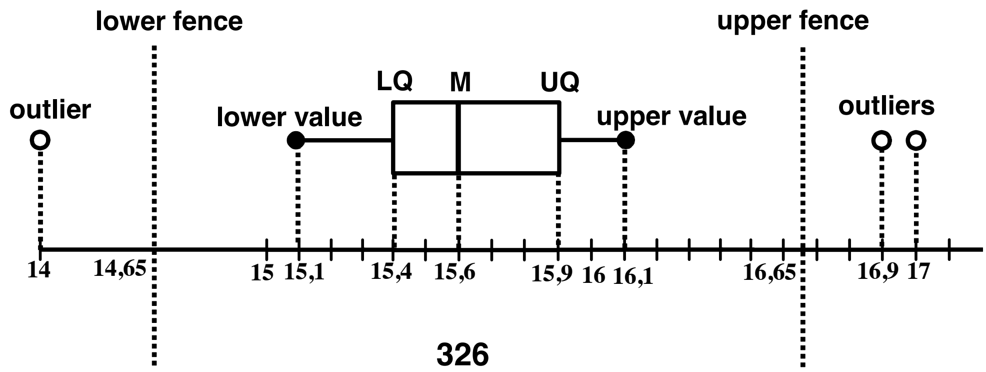
$$\text{Lower fence} = Q_1 - \frac{3}{2} \times \text{IQR} = 15{,}4 - \frac{3}{2} \times 0{,}5 = 14{,}65$$

$$\text{Upper fence} = Q_3 + \frac{3}{2} \times \text{IQR} = 15{,}9 + \frac{3}{2} \times 0{,}5 = 16{,}65$$

Any number outside the interval $[14{,}65\,;16{,}65]$ is an outlier.

14 is an outlier (less than 14,65)
16,9 and 17 are outliers (greater than 16,65)

(c) In Tukey's model, the outliers are indicated as open dots in the box and whisker plot. The lower and upper fences are indicated and the whiskers end at the lowest and highest data values that lie within the interval $\left[Q_1 - \frac{3}{2} \times \text{IQR}\,; Q_3 + \frac{3}{2} \times \text{IQR}\right]$. These values are referred to as the lower value and upper value respectively.



(d) Outliers are **included** in the data set:

14	15,1	15,4	15,4	15,4	15,5	15,5	15,6
15,7	15,7	15,7	15,9	16,1	**16,9**	**17,0**	

$\text{Mean} = \frac{234{,}9}{15} = 15{,}7$ Standard deviation $= 0{,}7$

$\text{M} = 15{,}6$

Outliers are **excluded** from the data set:

-	15,1	15,4	15,4	15,4	15,5	15,5	15,6
15,7	15,7	15,7	15,9	16,1	-	-	

$\text{Mean} \frac{187}{12} = 15{,}58$ Standard deviation $= 0{,}3$

$\text{M} = \frac{15{,}5 + 15{,}6}{2} = 15{,}55$

You will notice that the mean for the data including the outliers is different to the mean when the outliers are excluded. The standard deviations in each case are also different. The value of the median is only slightly affected by the exclusion of the outliers.

EXAMPLE 6

Consider the following set of numbers:

21 23 24 25 29 33 49

(a) Draw a box and whisker plot for this data.

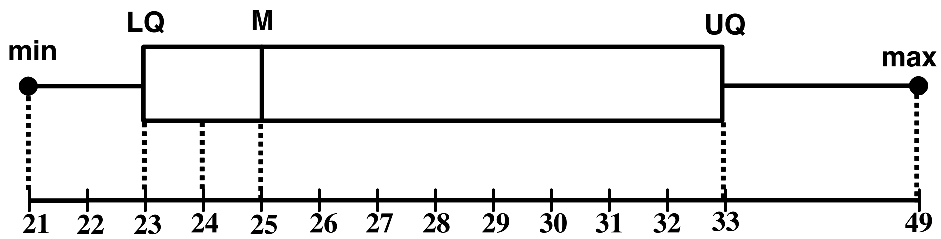
(b) Determine the outliers.

(c) Draw a box and whisker plot highlighting the outliers.

(d) Discuss the effect of the outliers on the mean, median and standard deviation if the outliers are included in the data set and if they are excluded.

Solution

(a) Min = 21 $Q_1 = 23$ M = 25 $Q_3 = 33$ Max = 49



(b) $\text{IQR} = 33 - 23 = 10$

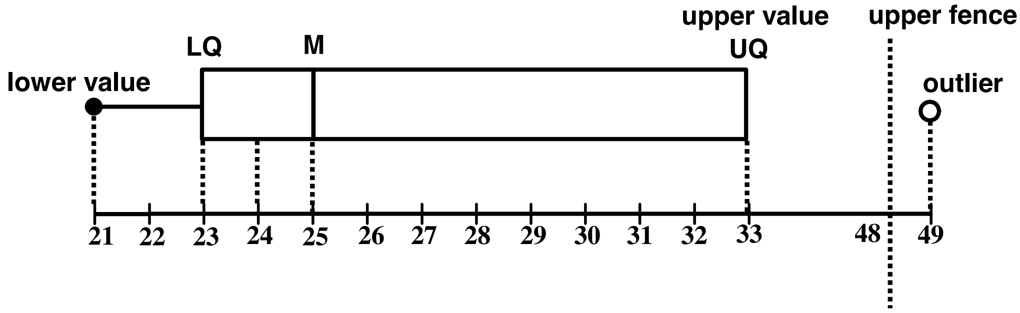
$\text{Lower fence} = Q_1 - \frac{3}{2} \times \text{IQR} = 23 - \frac{3}{2} \times 10 = 8$

$\text{Upper fence} = Q_3 + \frac{3}{2} \times \text{IQR} = 33 + \frac{3}{2} \times 10 = 48$

Any number outside the interval $[8\,;48]$ is an outlier.

49 is an outlier (greater than 48)

(c) In this box and whisker plot, the upper quartile 33 is also the upper value if we exclude the outlier 49. The upper whisker falls away.



(d) Outlier is **included** in the data set:

21 23 24 25 29 33 **49**

$\text{Mean} = \frac{204}{7} = 29{,}1$ Standard deviation $= 8{,}9$

$M = 25$

Outlier is **excluded** from the data set:

21 23 24 25 29 33 **-**

$\text{Mean} = \frac{155}{6} = 25{,}8$ Standard deviation $= 4{,}0$

$M = \frac{24 + 25}{2} = 24{,}5$

You will notice that the mean for the data including the outlier is significantly different to the mean when the outlier is excluded. The standard deviations in each case are significantly different. The value of the median is only slightly affected by the exclusion of the outliers.

<u>Discussion</u>

In research situations, whether or not outliers are included or excluded from the data depends on what the researcher wants from the Statistics. Outliers can adversely affect the mean and standard deviation and are often excluded because of this. They might also be excluded if they are meaningless for what is being researched. For example, if you want to know the average height of people in your city, you might want to exclude the few people who are very tall or very short. Sometimes, outliers can be errors in measurement and these values must be excluded. Some researchers prefer to include outliers in large data sets that are unusual but not necessarily errors in measurement. This is to ensure maximum accuracy of results.

EXERCISE 5

1. The earnings of the top twelve international sportsmen during 2011 is presented in the following table:

Position	Amount in millions of US dollars	Sport
1	75	Golf
2	53	Basketball
3	48	Basketball
4	47	Tennis
5	47	Golf
6	40	Football
7	38	Football
8	35	Baseball
9	34	Racing
10	32	Football
11	32	Racing
12	31	Tennis

Source: www.the richest.org/sports/forbes

(a) Which sportsman is considered to be an outlier? Show all workings to verify your answer.

(b) Draw a box and whisker plot of this data highlighting the outliers.

2. During the 2011 South African Premier League football season, the total number of points scored by twenty clubs is as follows:

Football Club	Goals scored
Pirates	43
Sundowns	30
Swallows	26
Chiefs	19
SuperSport	26
FS Stars	17
Ajax	26
AmaZulu	16
Celtic	15
Arrows	19
Plat Stars	14
Wits	13
Maritzburg	12
Leopards	12
Santos	15
Cosmos	15
Univ of Pta	13
Rovers	19
Thanda Royal	12
Vasco	17

Source: www.psl.co.za

(a) Which club is considered to be an outlier? Show all workings to verify your answer.

(b) Draw a box and whisker plot of this data highlighting the outliers.

(c) Discuss the effect of the outliers on the mean, standard deviation and median if the outliers are included in the data set and if they are excluded.

3. The earnings of the top twelve international celebrities during 2011 is presented in the following table:

Position	Amount in millions of US dollars	Field
1	290	TV
2	195	Music
3	130	Actor
4	125	Music
5	113	Film producer
6	107	Film producer
7	100	Music
8	90	Music
9	90	Music producer
10	84	Writer

Source: www.ranker.com

(a) Which celebrities are considered to be outliers? Show all workings to verify your answer.

(b) Draw a box and whisker plot of this data highlighting the outliers.

4. The number of matric maths papers marked by ten matric markers during a marking session is as follows:

50 95 100 115 155 40 90 95 105 120

(a) Determine the quartiles and the inter-quartile range.

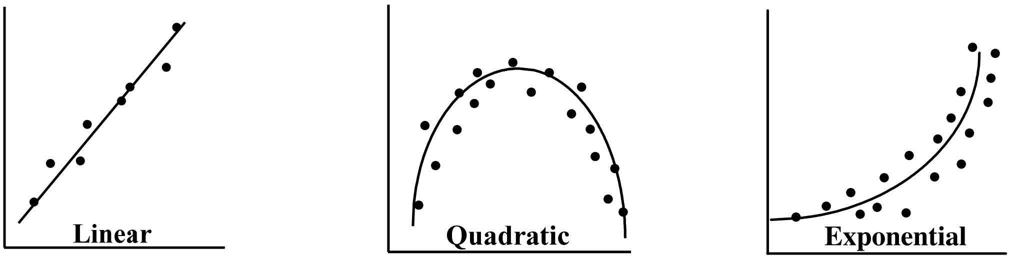
(b) Determine the outliers.

(c) Draw a box and whisker plot highlighting the outliers.

(d) What possible reasons can be given for having markers that are outliers?

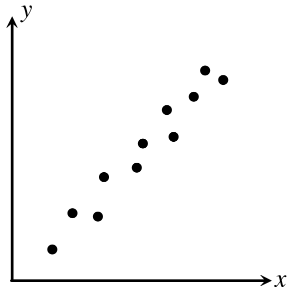
SCATTERPLOTS AND OUTLIERS

Sometimes there is a relationship between two sets of data. For example, the number of cars on a highway and the accident rate are two sets of data that might have a relationship or **correlation**. Plotting this data on a scatterplot diagram will show trends in the data. Data could follow a linear, quadratic or exponential trend.

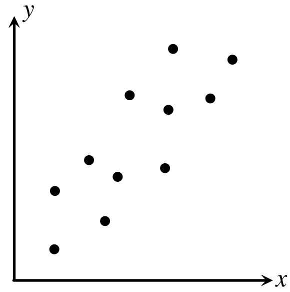


The strength of the linear relationship between the two variables in a scatter plot depends on how close the data points are to the line of best fit. The closer the points are to this line, the stronger the relationship. If the points are further away from the line of best fit, the weaker the relationship. If the line of best fit slopes to the right and has a positive gradient, then the linear relationship is positive. If the line of best fit slopes to the left and has a negative gradient, then the linear relationship is negative.

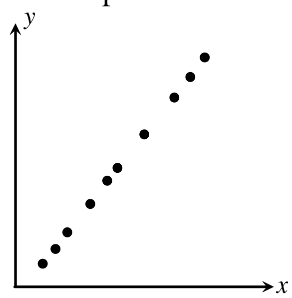
Strong positive linear association



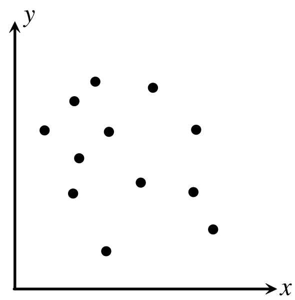
Moderate positive linear association



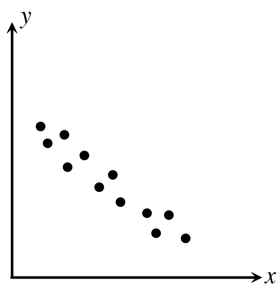
Perfect positive linear association



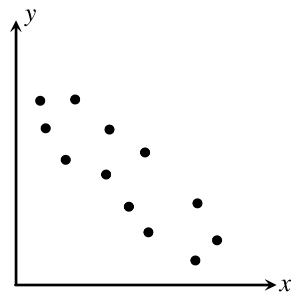
No positive correlation



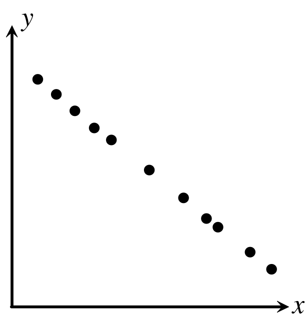
Strong negative linear association



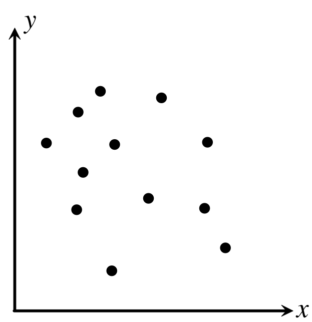
Moderate negative linear association



Perfect negative linear association



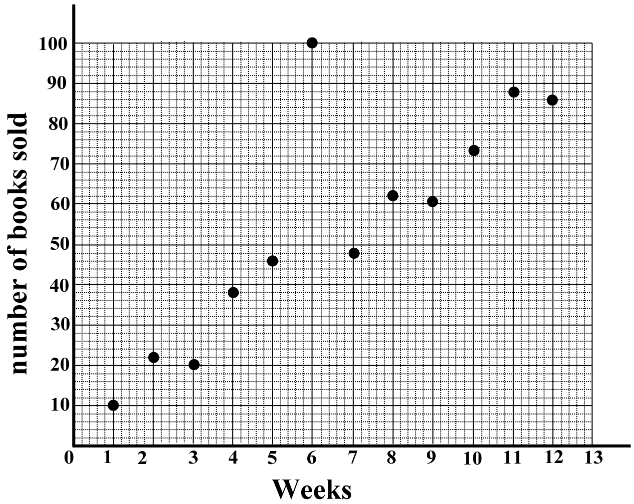
No negative correlation



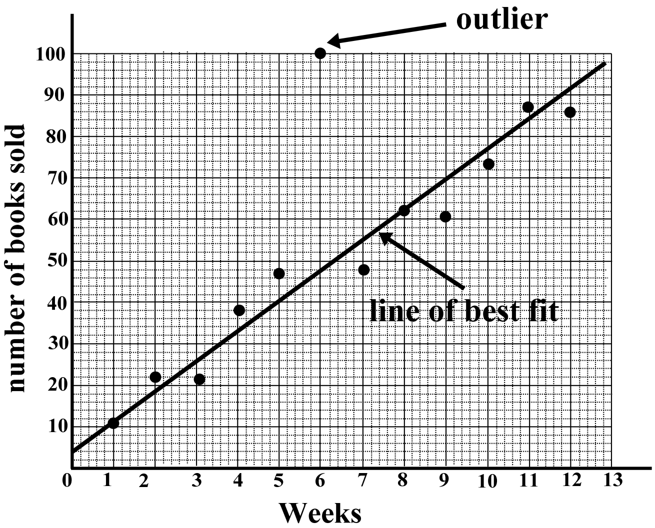
In Grade 12, we will explore the concept of correlation in more detail.

EXAMPLE 7

Consider the following scatterplot of information obtained by a publishing company which recorded the number of books sold per week.



It is clear from the above graph that the points do not lie on a perfect straight line. The information on the graph does seem to follow a linear trend. We can draw what is called a **line of best fit** which will help to predict future values.
In order to do this, draw a line through some of the points with the aim of having the same number of points above the line as below the line.
A possible line of best fit will now be drawn onto the graph.



A line can be drawn through two of the points with five above and five below. A line of best fit will not represent the data perfectly, but it will give you an idea of the trend. Different people will probably draw slightly different lines of best fit, but the trend should be more or less the same in each case. In Grade 12 you will determine the actual equation of the line of best fit using the least squares method. One of the points (see diagram) is way out compared to the others. This point is called an **outlier**. Probably what happened during the sixth week was that the company had a high sale of books. There might have been a conference where the company displayed and sold a lot of books.

EXERCISE 6 (Photocopyable grids are provided in the Teacher's Guide)

1. A nursery recorded the effect of temperature on the growth of a new plant that has recently been imported into the country. The goal of the study is to determine what temperature is ideal for maximum flowering for a particular plant.

Temperature	Number of flowers
25°C	2
26°C	3
27°C	16
28°C	6
29°C	7
30°C	7
31°C	8
32°C	9
33°C	10
34°C	12
39°C	15
40°C	1

(a) Draw a scatter plot to represent this data.

(b) Which temperatures are outliers? Explain possible causes for these outliers.

2. The table below represents the number of people infected with the TB virus in a certain city from the year 2006 to 2011.

Year	Number of people infected with the TB virus
2006	118
2007	123
2008	131
2009	134
2010	136
2011	138

(a) Draw a scatter plot to represent this data.

(b) Explain whether a linear, quadratic or exponential curve would be a line or curve of best fit.

(c) If the same trend continued, estimate, by using your graph, the number of people that will be infected in 2012.

3. The table below shows the acidity of eight lakes near an industrial plant and their distance from it.

Distance (km)	4	34	17	60	6	52	42	31
Acidity (pH)	3.0	4.4	3.2	7.0	3.2	6.8	5.2	4.8

(a) Draw a scatter plot to illustrate this data.

(b) Draw a line of best fit on the diagram.

(c) Use your line of best fit to predict the acidity of the lake at a distance of 22 kilometres.

(d) Describe the correlation between the distance and acidity.

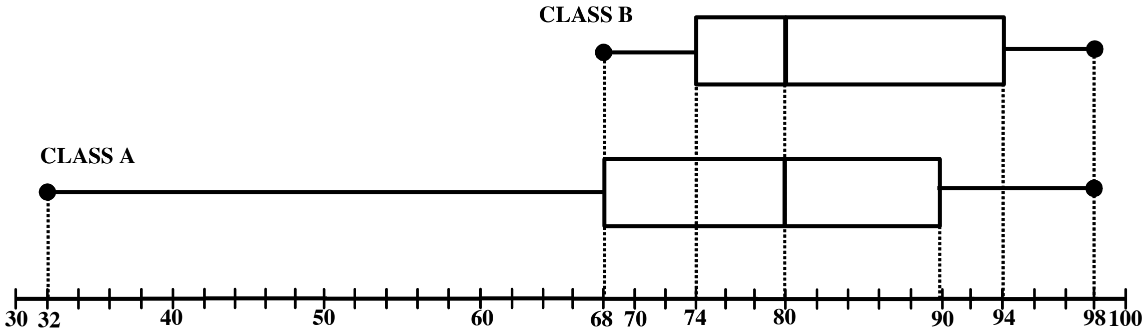
4. A medical researcher recorded the growth in the number of bacteria over a period of 10 hours. The results are recorded in the following table:

Time in hours	0	1	2	3	4	5	6	7	8	9	10	11
Number of bacteria	5	10	75	13	10	20	30	35	45	65	80	1

(a) Draw a scatter plot to represent this data.
(b) State the type of relationship (linear, quadratic or exponential) that exists between the number of hours and the growth in the number of bacteria.
(c) Are there any outliers? Explain possible causes for these outliers.

REVISION EXERCISE

1. The box and whisker plots below summarise the final examination percentages for two Accounting classes.



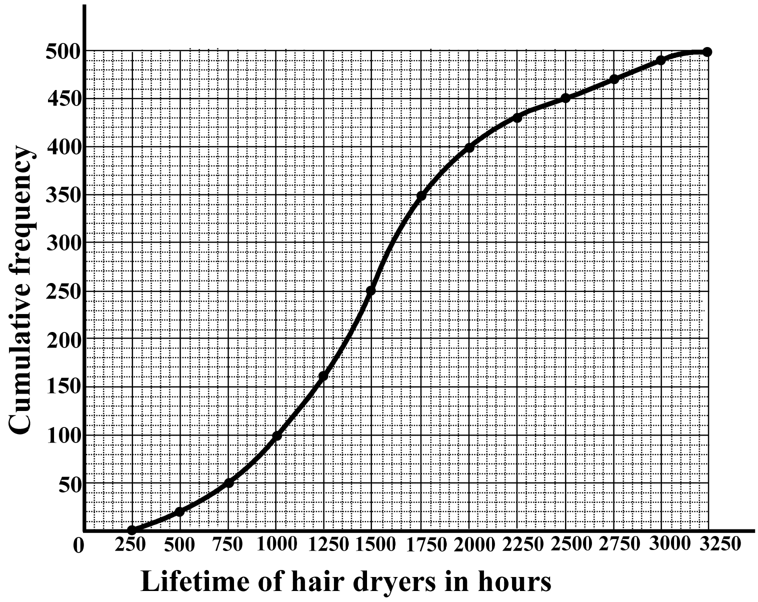
(a) What features are the same for both classes?
(b) The teacher considers the median of each class and reports that there is no significant difference in the performance between them. Is this conclusion valid? Support your answer with reasons.
(c) Comment on the distribution of marks for Class A.
(d) Determine whether the minimum or maximum values for Class A are outliers.
(e) Does Class B have any outliers? Explain.

2. During an athletics season, Mpho recorded his times in seconds for his performance in the 400m track event. His results for each event are as follows:

44,3	45,9	45,3	47,8	45,3	44,8	45,7	45,2	44,9	44,7
45,7	46,3	46,2	45,6	44,8	45,7	45,0	46,3	44,9	45,1
46,1	45,0	46,2	45,4	45,5	45,9	44,7	46,0	45,1	45,1

(a) Arrange the times from smallest to largest (Use a stem and leaf diagram).
(b) Calculate the mean for this data.
(c) Draw a box and whisker plot for this data.
(d) Comment on the distribution of the data.
(e) Calculate the standard deviation for this data (one decimal place).
(f) How many 400m times lie outside the first standard deviation interval?
(g) Which time is an outlier? Give a possible reason for this outlier.
(h) Redraw the box and whisker plot highlighting the outlier.

3. The lifetimes of electrical hair dryers were recorded by a company. The results are shown in the following cumulative frequency diagram.



(a) How many hair dryers were there?
(b) How many hair dryers had a lifetime of 750 hours or less?
(c) How many hair dryers had a lifetime of 750 hours or more?
(d) How many hair dryers had a lifetime of between 2000 and 3250 hours?
(e) After how many hours were 20% of the hair dryers dead?
(f) What was the shortest possible lifetime of a hair dryer?
(g) What percentage of hair dryers had a lifetime of between 750 and 1500 hours?
(h) What is the median time for the lifetime of a hair dryer?
(i) Redraw and complete the following table using the above graph.

Class interval	Frequency	Cumulative frequency
$0 < x \leq 250$		0
$250 < x \leq 500$		
$500 < x \leq 750$		
$750 < x \leq 1000$		
$1000 < x \leq 1250$		
$1250 < x \leq 1500$		
$1500 < x \leq 1750$		
$1750 < x \leq 2000$		
$2000 < x \leq 2250$		
$2250 < x \leq 2500$		
$2500 < x \leq 2750$		
$2750 < x \leq 3000$		
$3000 < x \leq 3250$		

(j) Draw a histogram and frequency polygon. Use the above table to assist you. (A grid is provided in the Teacher's Guide – this may be photocopied for the learners)
(k) Now calculate the estimated mean for this data.

(l) Use the cumulative frequency curve to draw a box and whisker plot for this data. You may assume that the minimum lifetime for a hair dryer is 250 and the maximum lifetime is 3250.

(m) Describe how the data is distributed. State a reason for your answer.

SOME CHALLENGES

1. The table below contains the mean, median, standard deviation and range of the Mathematics final exam for a large group of students.

Mean	Median	Standard deviation	Range
56	51	17,4	86

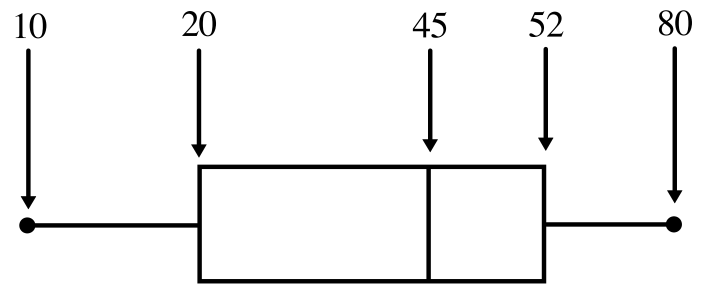
The Mathematics teacher added 3 marks to each of the students' marks. Write down the mean, median, standard deviation and range for the new set of Mathematics results.

2. Consider the following set of data values: $x\ ; 2x-1\ ; 2x\ ; 2x+2\ ; 3x-1$
The inter-quartile range is 6. Determine the value of x and hence the variance.

3. Consider the following set of data values: 2 ; 3 ; 5 ; x ; 16
If the mean is 7 and the standard deviation is 5,099019514, use two different methods to calculate the value of x.

4. Consider the following set of data values: 1 ; x ; 7 ; y ; 10
If the mean is 6 and the variance is 10, calculate the value of x and y.

5. Consider the following box and whisker plot:



The data set contains a total of nine numbers. The second and third numbers of the data set are the same. The seventh and eighth numbers are different. The eighth number is one more than the 75th percentile. The mean for the data set is 40.
Write down a possible list of nine numbers which will result in the above box and whisker plot.

SOLUTIONS TO EXERCISES

CHAPTER 1

EXERCISE 1

1(a) 6 **1(b)** $-8x^9y^6$ **1(c)** 3^8 **1(d)** $\frac{4}{3y^2}$ **1(e)** $\frac{16x^6y^4}{25}$ **1(f)** $\frac{4y^4}{25}$

1(g) $\frac{3b^5}{a^2}$ **1(h)** $81x^{12}+36x^6$ **1(i)** xy **2(a)** 2011 **2(b)** 512 **2(c)** $\frac{81}{2}$

2(d) 64 **2(e)** $\frac{27}{128}$ **2(f)** $16\frac{1}{16}$ **2(g)** $\frac{12}{7}$

EXERCISE 2

2(a) 1250 **2(b)** 81 **2(c)** $\frac{4}{3}$ **2(d)** $\frac{2}{3}$ **2(e)** 2 **2(f)** 2 **2(g)** $\frac{1}{2}$

2(h) 3 **2(i)** $32\frac{7}{15}$ **3(a)** 2^x **3(b)** 2^x-1 **3(c)** 3^x+3 **3(d)** $-4-2^x$

EXERCISE 3

(a) 3 **(b)** 8 **(c)** 3 **(d)** 9 **(e)** 8 **(f)** $\frac{1}{4}$ **(g)** $\frac{1}{9}$

(h) $\frac{25}{4}$ **(i)** $\frac{1}{2}$ **(j)** $\frac{2}{3}$

EXERCISE 4

1(a) $a^{\frac{3}{2}}$ **1(b)** $x^{\frac{5}{3}}$ **1(c)** $a^{\frac{5}{2}}$ **1(d)** $x^{\frac{3}{4}}$ **1(e)** $a^{\frac{2}{3}}$ **1(f)** $x^{\frac{1}{2}}$

2(a) $\sqrt[5]{a^4}$ **2(b)** $\sqrt[6]{x}$ **2(c)** $\sqrt{a^5}$ **2(d)** $\sqrt[3]{x^2}$ **2(e)** $\sqrt{a}$ **2(f)** $\sqrt{x^3}$

3(a) 16 **3(b)** 2 **3(c)** 9 **3(d)** $\frac{1}{8}$ **3(e)** 4 **3(f)** $\sqrt[3]{4}$

3(g) 9 **3(h)** a **3(i)** x **3(j)** $64x^{12}$

4(a) $\frac{1}{3125}$ **4(b)** 8 **4(c)** $8x^2$ **4(d)** $\frac{27}{8}$ **4(e)** $81k$ **4(f)** $\frac{1}{8}$ **5.** 6

EXERCISE 5

1(a) $\sqrt{21}$ **1(b)** 18 **1(c)** 12 **1(d)** $9\sqrt{6}$ **1(e)** $3\sqrt[4]{3}$ **1(f)** $4\sqrt{3}$

1(g) $2\sqrt{2}$ **1(h)** 4 **1(i)** $\sqrt[3]{17}$ **1(j)** $2\sqrt{2}$ **1(k)** $\frac{\sqrt{6}}{3}$ **1(l)** 3

1(m) 2 **1(n)** -1 **1(o)** 2 **1(p)** 22 **1(q)** $28-10\sqrt{3}$ **1(r)** $53-10\sqrt{6}$

1(s) -50 **1(t)** $8\sqrt{15}$ **1(u)** $-64m$ **1(v)** 8 **2(a)** $32\sqrt{2}$ **2(b)** $8\sqrt[3]{4}$

2(c) $9x^3\sqrt{5x}$ **3.** $2\sqrt{2}$ **4.** -1 **5.** $\frac{1}{2}$ **6.** $\sqrt{3}-\sqrt{2}$

EXERCISE 6

1(a) $\frac{\sqrt{6}}{2}$ **1(b)** $\frac{2\sqrt{6}}{3}$ **1(c)** $\frac{\sqrt{2}}{5}$ **1(d)** $\frac{3}{2}$ **1(e)** $\sqrt[3]{3}$ **2(a)** $-1-\sqrt{3}$

2(b) $\frac{4\sqrt{7}+8}{3}$ **2(c)** $\frac{4-\sqrt{2}}{7}$

EXERCISE 7

(a) 0 **(b)** $\frac{1}{2}$ **(c)** $\frac{3}{2}$ **(d)** $-\frac{3}{2}$ **(e)** $\frac{1}{2}$ **(f)** $-1\frac{1}{2}$ **(g)** $\frac{1}{2}$

(h) -1 **(i)** $\frac{4}{7}$ **(j)** 0 **(k)** $\frac{2}{3}$ **(l)** 2 **(m)** $\frac{1}{5}$ **(n)** $\frac{1}{5}$

EXERCISE 8

1(a) 1 **1(b)** 1 **1(c)** 4 **1(d)** 1 **1(e)** 2 **1(f)** $0\,;1$ **1(g)** 0

1(h) -2 **1(i)** 0 **1(j)** -1 **1(k)** $2\,;1$ **1(l)** $0,5$ **1(m)** -2 **1(n)** $3\,;1$

2(a) $\frac{21}{3^x}$ **2(b)** -1 **3(a)** $5\left(\sqrt{2}\right)^x$ **3(b)** -2

EXERCISE 9

1(a) 9 **1(b)** none **1(c)** 27 **1(d)** -27 **1(e)** 16 **1(f)** none **1(g)** ± 512

1(h) none **1(i)** 25 **1(j)** none **1(k)** 32 **1(l)** -32 **1(m)** $\frac{9}{16}$ **1(n)** $\pm\frac{1}{64}$

1(o) $\frac{1}{9}$ **1(p)** ± 8 **1(q)** 3 **1(r)** $\frac{1}{729}$ **2(a)** $16\,;1$ **2(b)** $4\,;1$ **2(c)** 81

2(d) $\frac{27}{64}\,;-8$ **2(e)** $\frac{1}{9}$ **2(f)** ± 27

EXERCISE 10

(a) 29 **(b)** none **(c)** 6 **(d)** $2\,;1$ **(e)** $-3\,;-1$ **(f)** 7 **(g)** 1

(h) -4 **(i)** 4

REVISION EXERCISE

1(a) $\sqrt{2}$ **1(b)** $\frac{17}{2}$ **1(c)** 6 **1(d)** $-2-4\sqrt{2}$ **1(e)** $\frac{11}{x^3}$ **1(f)** $2x$ **1(g)** $-2\sqrt{2}$

2. $\frac{\sqrt{3}}{6}$ **3.** $\frac{1}{6}$ **4(a)** 729 **4(b)** -2 **4(c)** 27 **4(d)** 256 **4(e)** 8

4(f) ± 8 **5(a)** 0 **5(b)** $\frac{1}{2}$ **6(a)** 0 **6(b)** -2 **6(c)** $-125\,;64$ **6(d)** $1\,;0$

7(a) $x^{\frac{3}{2}}\left(2x-1\right)$ **7(b)** $\frac{1}{2}\,;1$

SOME CHALLENGES

1. $\frac{13}{3^x}$ **2.** $\sqrt{3}$ **3.** $x(x-1)^2$ **4.** $32\sqrt{2}$ **5.** (3) **6.** 6

7(a) 3^{360} **7(b)** $\sqrt[5]{16}$ **8.** 2 **9.** $-\frac{1}{2^8}$ **10(a)** $1{,}6\times 10^{2008}$ **10(b)** 7

11(a)(1) $T_1 = x^{\frac{1}{2}}$ **11(a)(2)** $x^{\frac{3}{4}}$ **11(a)(3)** $x^{\frac{15}{16}}$ **11(b)(1)** $x^{\frac{1023}{1024}}$

11(b)(2) $x^{\frac{2^n-1}{2^n}}$ **11(b)(3)** $x^{\frac{2^{100}-1}{2^{100}}}$ **11(b)(4)** $T_1 = x^1$ **12.** $\frac{5}{3}\,;\frac{2}{3}$

CHAPTER 2

EXERCISE 1

(a) $(x-2)^2-4$ **(b)** $\left(x+\frac{5}{2}\right)^2-\frac{25}{4}$ **(c)** $(x-3)^2-9$ **(d)** $(x-3)^2-1$

(e) $(x-4)^2-6$ **(f)** $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ **(g)** $\left(x+\frac{3}{2}\right)^2-\frac{41}{4}$ **(h)** $(x-5)^2-27$

EXERCISE 2

1(a) $2(x-2)^2-2$ **1(b)** $3(x-1)^2+3$ **1(c)** $-(x+1)^2-1$ **1(d)** $2\left(x-\frac{5}{4}\right)^2+\frac{7}{8}$

2. $2\left(x+\frac{3}{2}\right)^2+5\frac{1}{2}$ **3.** $-(x-2)^2-5$

EXERCISE 3

1(a) Max of 0 **1(b)** Min of 0 **1(c)** Max of 0 **1(d)** Min of 1 **1(e)** Min of 0

1(f) Max of 9 **1(g)** Min of -3 **2.** 4 **3.** $-\frac{9}{2}$ **4.** $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

5(a) $100x-x^2$ **5(b)** $50m\,;50m$

EXERCISE 4

(a) $6\,;-1$ **(b)** $0\,;1$ **(c)** $-12\,;-1$ **(d)** $8\,;1$ **(e)** $-5\,;4$

(f) $\frac{3}{2}\,;1$ **(g)** $2\,;-1$ **(h)** $4\,;-4$ **(i)** $3\,;-3$ **(j)** $\pm\sqrt{6}$

(k) $\pm\sqrt{7}$ **(l)** $\pm\sqrt{7}$ non-real

EXERCISE 5

1(a) $-2\pm\sqrt{10}$ **1(b)** $\frac{-1\pm\sqrt{5}}{2}$ **1(c)** $\frac{5\pm\sqrt{17}}{2}$ **1(d)** $-2\frac{1}{2}\,;1$ **1(e)** $\frac{7\pm\sqrt{13}}{6}$

1(f) $3\pm\sqrt{17}$ **1(g)** $\frac{5\pm\sqrt{105}}{4}$ **2.** $\frac{b+a}{a}\,;-1$ **3.** none

EXERCISE 6

1(a) $\frac{7\pm\sqrt{85}}{2}$ **1(b)** $\frac{9\pm\sqrt{57}}{6}$ **1(c)** $-\frac{1}{2}\,;1$ **1(d)** $2\,;-\frac{9}{2}$ **1(e)** none

1(f) none **2(a)** none **2(b)** $0{,}3\,;6{,}7$ **2(c)** none **2(d)** $2{,}3\,;-1{,}9$

2(e) $-0{,}1\,;2{,}4$

EXERCISE 7

(a) $6\,;-3$ **(b)** $13\,;-1$ **(c)** reals **(d)** none **(e)** $0\,;-\frac{24}{5}$

(f) 4 **(g)** $4\,;-4$ **(h)** reals **(i)** $-75\,;1$ **(j)** $1{,}6;-0{,}6$

EXERCISE 8

1(a) non-real **1(b)** real, irrational, unequal **1(c)** real, rational, unequal

1(d) real, rational, equal **1(e)** real, rational, unequal **1(f)** real, irrational, unequal

2. $\Delta=0$ **3.** $\Delta=-11m^2$ **4(a)** $\Delta=9a^2$ **4(b)** equal **5.** $\Delta=p^2$ **6.** $\Delta=4r^2+12$

7(a) $k=-\frac{25}{4}$ **7(b)** $k\geq-\frac{25}{4}$ **7(c)** $k<-\frac{25}{4}$ **8.** $k=3$ **9(a)** $p\leq\frac{9}{16}$ **9(b)** $p>\frac{9}{16}$

10. $k=1;\ 4$ **11.** $k>-1$ **12(a)** $m=-2$ **12(b)** $m=\frac{9}{20}$ **12(c)** $m\leq\frac{9}{20}$

13(a) $k\leq\frac{21}{4}$ **13(b)** $k=3$

EXERCISE 9

1(a) $x=4\,;y=2$ **1(b)** $x=4\,;y=10$ **1(c)** $x=1\,;y=1$ **1(d)** $x=5\,;y=-2$ **2.** $(-1\,;-7)$

3(a) $x=1$ or $x=-1\,;y=4$ or $y=0$ **1(b)** $x=0$ or $x=1\,;y=-3$ or $y=-4$

3(c) $x=\frac{2}{3}$ or $x=-\frac{5}{3}\,;y=0$ or $y=-7$ **3(d)** $x=2{,}2$ or $x=-1{,}8\,;y=2{,}1$ or $y=0{,}1$

3(e) $x=\frac{1}{3}$ or $x=\frac{9}{2}\,;y=-3$ or $y=\frac{19}{2}$ **3(f)** $x=\frac{5}{2}$ or $x=-1\,;y=-\frac{1}{4}$ or $y=5$

3(g) $x=4$ or $y=-2$ **3(h)** $x=-\frac{5}{2}$ or $x=4\,;y=11$ or $y=-2$

3(i) $x=9$ or $x=-1\,;y=3\frac{1}{3}$ or $y=0$ **4(a)** $a=4b$ or $a=b$

4(b) $4\,;1$ **4(c)** $a=1$ or $a=\frac{8}{5}\,;b=1$ or $b=\frac{2}{5}$

EXERCISE 10

1. $\pm18\,;\pm19$ **2.** 15m by $2{,}\dot{6}$m ; 8m by 5m **3.** 10m by 9,6m

4. 60km/h **5.** 16

EXERCISE 11

(a) $x\leq-3$ **(b)** $x>-8$ **(c)** $x<1$ **(d)** $x\geq5$ **(e)** $y\leq-\frac{1}{2}$ **(f)** $y>-6$

EXERCISE 12

1(a) $-4\leq x\leq4$ **1(b)** $x\leq-5$ or $x\geq5$ **1(c)** $-3<x<3$ **1(d)** $x<-1$ or $x>1$

1(e) $x<-1$ or $x>1$ **1(f)** $-3<x<4$ **1(g)** $-1\leq x\leq6$ **1(h)** $x<-\frac{3}{2}$ or $x>\frac{3}{2}$

1(i) $-\frac{1}{2}<x<3$ **1(j)** $x\leq-3$ or $x\geq\frac{1}{2}$ **1(k)** $x\leq0$ or $x\geq7$ **1(l)** $-1\leq x\leq3$

1(m) $-\frac{3}{2}\leq x\leq3$ **1(n)** reals **1(o)** none **1(p)** $x<-4$ or $x>-4$ **1(q)** reals

1(r) $x=-4$ **1(s)** none **1(t)** reals **1(u)** $-\sqrt{5}\leq x\leq\sqrt{5}$ **1(v)** $\{1\,;2\,;3\}$

2(a) $x<0$ **2(b)** $x<3$ **2(c)** $-1<x<0$ **3.** $x\leq-5$ or $x\geq5$

4(b)(1) reals **4(b)(2)** none

REVISION EXERCISE

1(a) $8;-1$ **1(b)** $5,56;1,44$ **1(c)** $0;\frac{4}{9}$ **1(d)** $-\frac{3}{2};-5$

1(e) $x \leq -8$ or $x \geq 8$ **1(f)** reals **1(g)** $-24,5;0,5$ **1(h)** none

1(i) $x < 0$ or $x > \frac{1}{2}$ **4.** real, irrational, unequal **5.** $0 < k \leq \frac{4}{3}$ **6(a)** $7;4$

6(b) $5,46;-1,46;5;-1$ **7(a)** $3(x-1)^2+9$ **7(b)** reals **8.** $2 < x < 3$

9. $x=-\frac{18}{5}$ or $x=6$; $y=\frac{6}{5}$ or $y=6$ **10.** 6 **11(c)** $-3 < x \leq 1$

SOME CHALLENGES

1(b) 31,38 **2.** 4 **3.** $\sqrt{2}$ **4(a)** $-\frac{4}{3}\left(x-\frac{3}{2}\right)^2+3\ 3$ **4(b)** 3

5(a) $4 < t < 12$ **;** $24 < t < 36$ **5(b)** 8km **6(a)** 0,8km **6(b)** 2,4km

CHAPTER 3

EXERCISE 1

1(a) $T_n = n+2$; $T_{100}=102$ **(b)** $n=50$ **2(a)** $T_n=2n+1$; $T_{180}=361$ **(b)** $n=120$

3(a) $T_n=5n-7$; $T_{259}=1288$ **(b)** $n=152$ **4(a)** $T_n=-2n+6$; $T_{150}=-294$ **(b)** $n=400$

5(a) $T_n=-4n-2$; $T_{600}=-2402$ **(b)** $n=110$ **6(a)** $T_n=\frac{1}{2}n+1$; $T_{130}=66$ **(b)** $n=80$

EXERCISE 2

1(b) $46;61$ **(c)** $T_n=n^2+4n+1$; $T_{100}=10401$ **(d)** $n=9$

2(b) $94;137$ **(c)** $T_n=4n^2-n-1$; $T_{160}=102239$ **(d)** $n=10$

3(b) $-73;-106$ **(c)** $T_n=-3n^2+2$; $T_{80}=-19198$ **(d)** $n=50$

4(a) $T_n=n^2+3$; $T_{150}=25503$ **4(b)** $T_n=n^2-1$; $T_{150}=22499$ **4(c)** $T_n=-2n^2+4$; $T_{150}=-44996$

4(d) $T_n=4n+2$; $T_{150}=302$ **4(e)** $T_n=3n^2$; $T_{150}=67500$ **4(f)** $T_n=5n^2+2n$; $T_{150}=112800$

4(g) $T_n=2n^2+n+\frac{1}{2}$; $T_{150}=45150\frac{1}{2}$ **4(h)** $T_n=\frac{n^2+1}{n^2+2}$; $T_{150}=\frac{22501}{22502}$

5(a) $x=17$ **(b)** $T_n=2n^2-1$ **6(a)** $x=-14$ **(b)** $T_n=-2n^2-4n+2$

7(a) $x=5$ and $y=13$ **(b)** $n=12$ **8(a)** $10100cm^2$ **(b)** $n=15$

REVISION EXERCISE

1(a) $T_n=3n+2$; $T_{16}=50$ **1(b)** $T_n=n^2+3n-1$; $T_{16}=303$ **1(c)** $T_n=4n-2$; $T_{16}=62$

1(d) $T_n=n^2+n$; $T_{16}=272$ **2(a)** $T_n=-2n+3$ **2(b)** -47 **2(c)** $T_n=-n^2+4n-7$

2(d) $n=10$ **3.** $10;2$ **4(a)** $S_n=\frac{1}{2}n^2+\frac{1}{2}n$ **4(b)** $S_{200}=20100$

5(a) $x=13$ **5(b)** 211

SOME CHALLENGES

1(a) 91 **1(b)** $T_n=3(n-1)^2+3(n-1)+1$ **2(a)** $31;48;69$ **2(b)(1)** $c=3$

2(b)(2) $d=2t^2-t+3$ **2(c)** $193m$

CHAPTER 4

REVISION EXERCISE

1(a) $3\sqrt{2}$ **1(b)** $3\sqrt{10}$ **1(c)** 13 **1(d)** 6 **2(a)** $17,4$

3(a) $\left(\frac{-1}{2};\frac{-11}{2}\right)$ **3(b)** $\left(\frac{-3}{2};\frac{3}{2}\right)$ **3(c)** $\left(\frac{1}{2};7\right)$ **3(d)** $(2;-3)$ **4(a)** 1 **4(b)** $-\frac{1}{3}$

4(c) undefined **4(d)** 0 **4(e)** $\frac{2}{p}$

EXERCISE 1

1(a) $0;8$ **1(b)** 6 **2(a)** $-10;11$ **2(b)** -13 **3.** 2

5(a) $a=\frac{3}{2}$ $k=-2$ **5(b)** $(-3;0)$

EXERCISE 2

1(a) AB||CD **1(b)** AB $\perp$ CD **1(c)** AB $\perp$ CD **1(d)** AB $\perp$ CD **2(a)** $\frac{3}{5}$

2(b) 4 **2(c)** 1 **3(a)** 13 **3(b)** $-\frac{23}{4}$ **3(c)** -17

EXERCISE 3

1(a) $21,8°$ **1(b)** $127,87°$ **1(c)** $90°$ **1(d)** $123,69°$ **1(e)** $53,13°$

2(a) -1 **2(b)** 1 **2(c)** $\sqrt{3}$ **2(d)** $-\frac{1}{\sqrt{3}}$ **2(e)** undefined

3. $126,87°\,;\,45°$

EXERCISE 4

1(a) $69,62°$ **1(b)** $80,41°$ **1(c)** $82,88°$ **1(d)** $71,99°$ **2.** $25,75°$

EXERCISE 5

1(a) $y=-3x-1$ **1(b)** $y=\frac{2}{3}x-8$ **1(c)** $y=-x-4\frac{1}{2}$ **1(d)** $x=-8$ **1(e)** $y=7$

2(a) $y=-2x-9$ **2(b)** $y=-2x+8$ **2(c)** $y=-\frac{1}{4}x+\frac{5}{4}$ **2(d)** $x=-1$ **2(e)** $y=3$

2(f) $y=-\frac{8}{5}x+\frac{24}{5}$ **3(a)** $\left(3;\frac{7}{2}\right);(-1;2)$ **3(b)** $y=\frac{11}{12}x+\frac{3}{4}$; $y=-\frac{1}{6}x+\frac{11}{6}$

3(c) $\left(1;\frac{5}{3}\right)$

EXERCISE 6

1(a) $y=-\frac{2}{3}x+\frac{1}{3}$ **1(b)** $y=\frac{1}{2}x+3$ **1(c)** $y=\frac{1}{2}x-\frac{3}{2}$ **1(d)** $y=-\frac{1}{3}x+6$ **1(e)** $y=-x+3$

1(f) $y=\sqrt{3}x-4$ **2(a)** $x=2$ **2(b)** $y=-2$ **3(a)** $y=-\frac{2}{3}x-\frac{5}{6}$ **3(b)** $y=2x+7$

3(c) $y=\frac{4}{3}x-\frac{2}{3}$ **3(d)** $y=-0,5x-1$ **4(a)** $56,31°$ **4(b)** 0,6

EXERCISE 7

(a) $(1;6)$ **(b)(1)** $(1;1)$ **(b)(2)** $(-1;4)$ **(b)(3)** $(2;3)$

EXERCISE 8

3. $(0;0)$ **4(a)** $(3;5)$ **5(a)** $y=\frac{3}{2}x-\frac{1}{2}$ **5(b)** $y=\frac{2}{3}x+\frac{5}{3}$ **5(c)** $\left(-4;\frac{13}{3}\right)$

6(a) $\frac{1}{2}$ **6(b)** $\left(\frac{7}{2};\frac{7}{2}\right)$ **6(c)** $(8;3)$ **6(d)** $y=\frac{1}{2}x-1$ **6(e)** $126,87°$

6(f) $100,3°$

REVISION EXERCISE

1(a) 5 **1(b)** $147,7°$ **2(a)** 3 **2(b)** $y=3x-6$ **2(c)** 4

2(d) $63,43°$ **2(e)** $y=x-12$ **3.** $y=2x$ **4.** 3 **5.** $y=-\sqrt{3}x+2$

6(b) $y=2x+3$ **7(a)** $(8;5)$ **7(b)** $y=-3x+19$ **7(c)** $34,7°$ **8(a)** $5\sqrt{2}\,;\,\sqrt{10}$

8(c)(1) $135°$ **8(c)(2)** $36,9°$ **8(d)** 15 **8(e)** $y=-\frac{1}{2}x+\frac{1}{2}$ **8(f)** $(3;-1)$

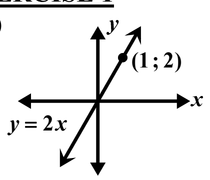
SOME CHALLENGES

1. F(5,91 ; 5) **2(b)** $33,9°$ **2(c)** $-0,2$ **3.** 6,5

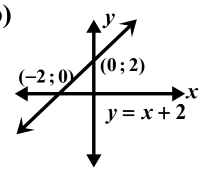
CHAPTER 5

EXERCISE 1

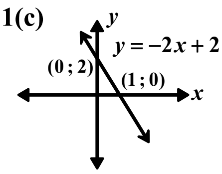
1(a)



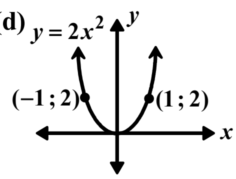
1(b)



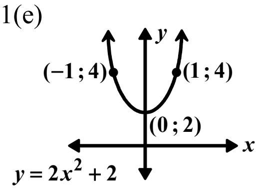
1(c)



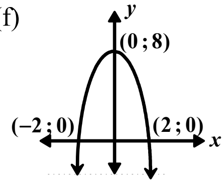
1(d)



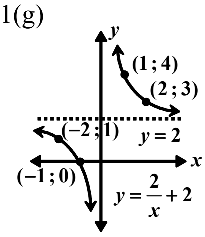
1(e)



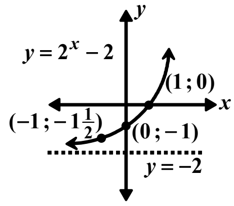
1(f)



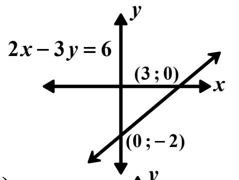
1(g)



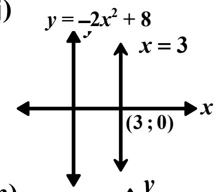
1(h)



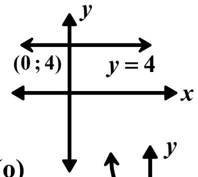
1(i)



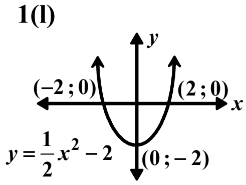
1(j)



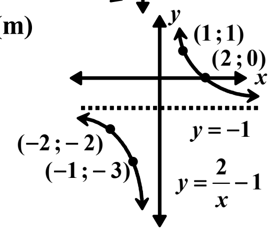
1(k)



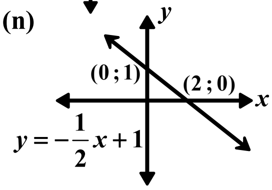
1(l)



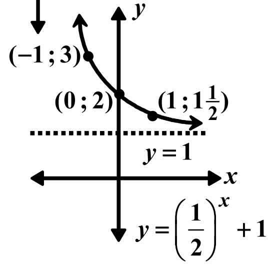
1(m)



1(n)

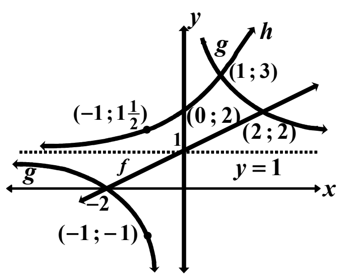


1(o)



2(a) $(-2\,;0)$ **;** $(0\,;1)$ **2(b)** g and h **2(c)** g $x = 0$

2(d)



2(e) $x \in (-\infty\,;\infty)$ $x \neq 0$

2(f) $y \in (1\,;\infty)$

2(g) f and h

3(a) $a > 0$ $q > 0$ **3(b)** $a < 0$ $q < 0$

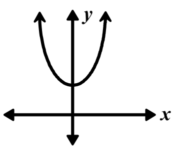
3(c) $a > 0$ $q < 0$ **3(d)** $a > 0$ $q > 0$

3(e) $a < 0$ $q < 0$ **3(f)** $a < 0$ $q = 0$

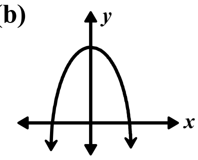
3(g) $a > 0$ $b > 0$ $q > 0$

3(h) $a > 0$ $b > 0$ $q > 0$ **3(i)** $a = 0$ $q > 0$

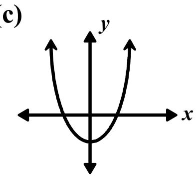
4(a)



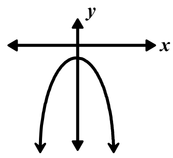
4(b)



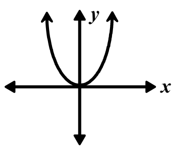
4(c)



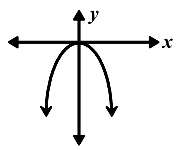
4(d)



4(e)



4(f)



5(a) $y = \frac{1}{2}x + 4$ **5(b)** $y = \left(\frac{1}{2}\right)^x + 3$

5(c) $y = \frac{-6}{x} + 3$ **5(d)** A$(-2\,;0)$

5(e) $y = -x^2 + 4$

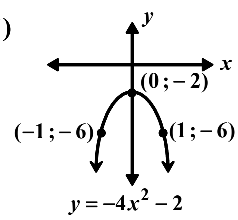
5(f) domain of f: $x \in (-\infty\,;\infty)$ range of f: $y \in (-\infty\,;\infty)$ **5(g)** $x > 0$

domain of g: $x \in (-\infty\,;\infty)$ range of g: $y \in (3\,;\infty)$ **5(h)** $y = x^2 - 4$

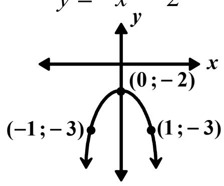
domain of h: $x \in (0\,;\infty)$ range of h: $y \in (-\infty\,;3)$ **5(i)** $y = 2^x + 3$

domain of j: $x \in (-\infty\,;\infty)$ range of j: $x \in (-\infty\,;4]$ **5(j)** $h(2x) - 6 = -4x^2 - 2$

5(j)

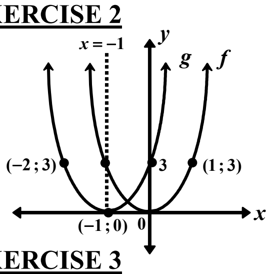


6. $y = -x^2 - 2$

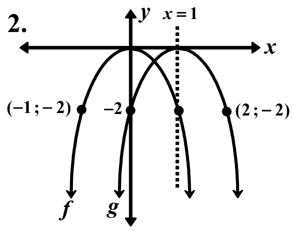


EXERCISE 2

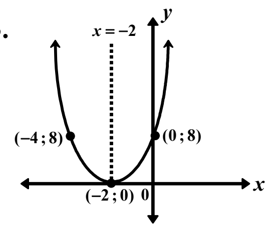
1.



2.

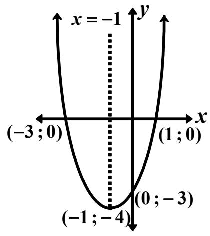


3.



EXERCISE 3

1(a)(1)



1(a)(2) $x \in (-\infty\,;\infty)$

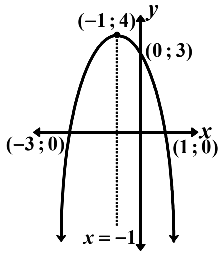
$y \in [-4\,;\infty)$

1(a)(3) Increases for $x > -1$

Decreases for $x < -1$

1(a)(4) Minimum value: –4

1(b)(1)



1(b)(2) $x \in (-\infty\,;\infty)$

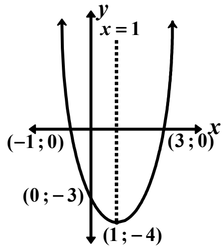
$y \in (-\infty\,;4]$

1(b)(3) Increases for $x < -1$

Decreases for $x > -1$

1(b)(4) Maximum value: 4

1(c)(1)



1(c)(2) $x \in (-\infty\,;\infty)$

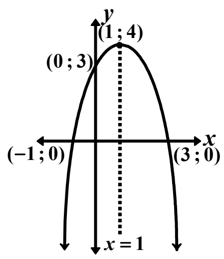
$y \in [-4\,;\infty)$

1(c)(3) Increases for $x > 1$

Decreases for $x < 1$

1(c)(4) Minimum value: –4

1(d)(1)



1(d)(2) $x \in (-\infty\,;\infty)$

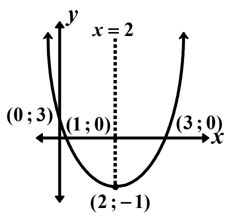
$y \in (-\infty\,;4]$

1(d)(3) Increases for $x < 1$

Decreases for $x > 1$

1(d)(4) Maximum value: 4

1(e)(1)



1(e)(2) $x \in (-\infty\,;\infty)$

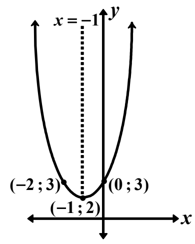
$y \in [-1\,;\infty)$

1(e)(3) Increases for $x > 2$

Decreases for $x < 2$

1(e)(4) Minimum value: –1

1(f)(1)



1(f)(2) $x \in (-\infty\,;\infty)$

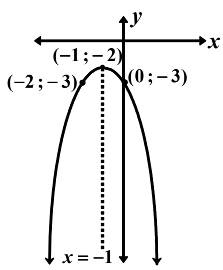
$y \in [2\,;\infty)$

1(f)(3) Increases for $x > -1$

Decreases for $x < -1$

1(f)(4) Minimum value: 2

1(g)(1)



1(g)(2) $x \in (-\infty\,;\infty)$

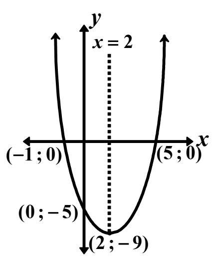
$y \in (-\infty\,;-2]$

1(g)(3) Increases for $x < -1$

Decreases for $x > -1$

1(g)(4) Maximum value: –2

1(h)(1)



1(h)(2) $x \in (-\infty\,;\infty)$

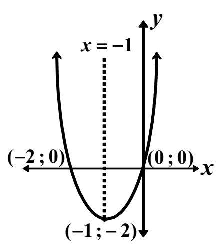
$y \in [-9\,;\infty)$

1(h)(3) Increases for $x > 2$

Decreases for $x < 2$

1(h)(4) Minimum value: –9

1(i)(1)



1(i)(2) $x \in (-\infty\,;\infty)$

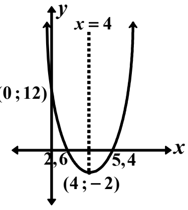
$y \in [-2\,;\infty)$

1(i)(3) Increases for $x > -1$

Decreases for $x < -1$

1(i)(4) Minimum value: –2

1(j)(1)



1(j)(2) $x \in (-\infty\,;\,\infty)$

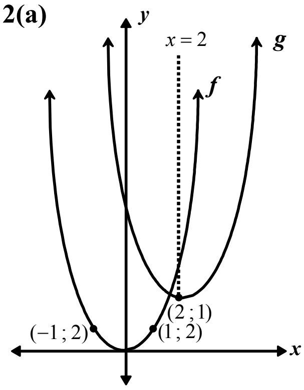
$y \in [-2\,;\,\infty)$

1(j)(3) Increases for $x > 4$

Decreases for $x < 4$

1(j)(4) Minimum value: −2

2(a)



3(a) $g(x) = (x+2)^2 - 4$

3(b) $g(x) = (x-2)^2 - 4$

3(c) $g(x) = x^2$

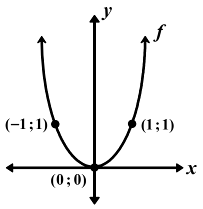
3(d) $g(x) = x^2 - 5$

3(e) $g(x) = (x+2)^2 - 1$

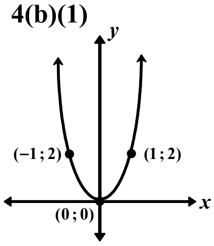
3(f) $g(x) = (x-2)^2 - 6$

3(g) $y = -x^2 + 4$

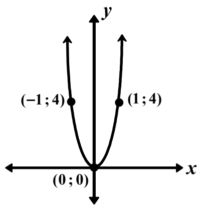
4(a)



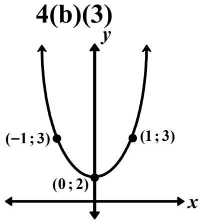
4(b)(1)



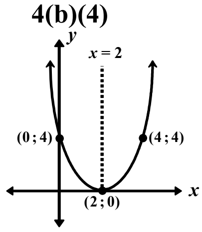
4(b)(2)



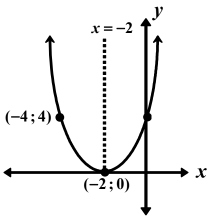
4(b)(3)



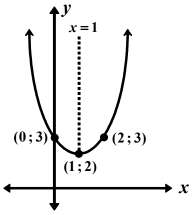
4(b)(4)



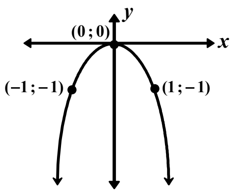
4(b)(5)



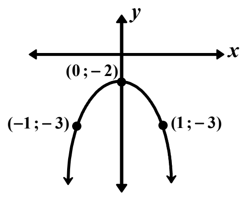
4(b)(6)



4(b)(7)

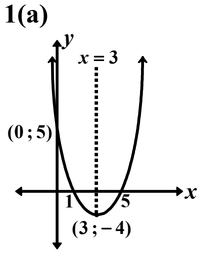


4(b)(8)

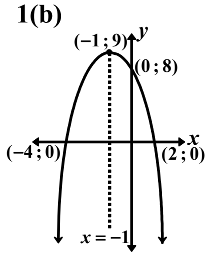


EXERCISE 4

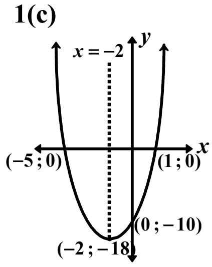
1(a)



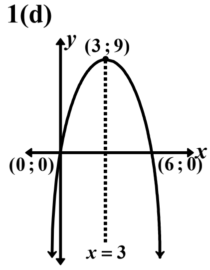
1(b)



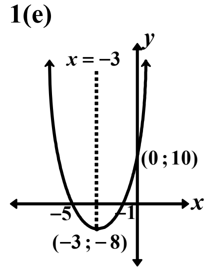
1(c)



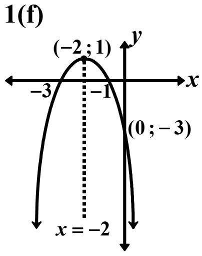
1(d)



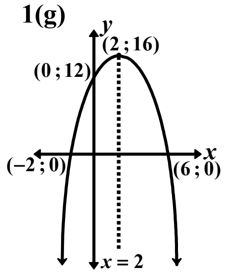
1(e)



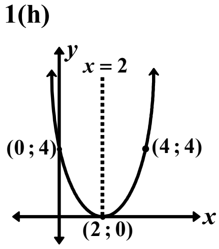
1(f)



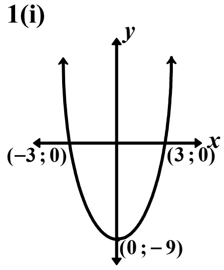
1(g)



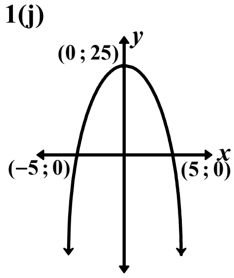
1(h)



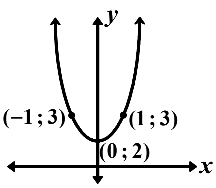
1(i)



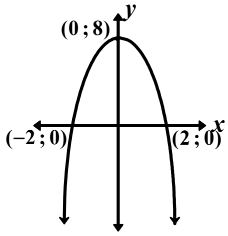
1(j)



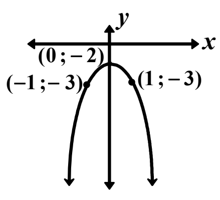
1(k)



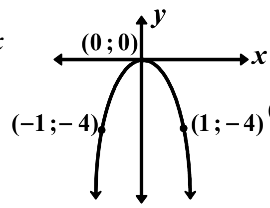
1(l)



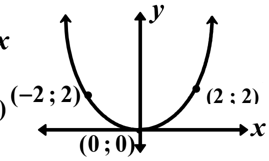
1(m)

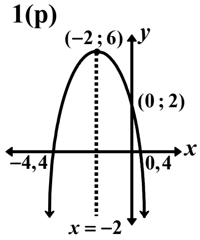


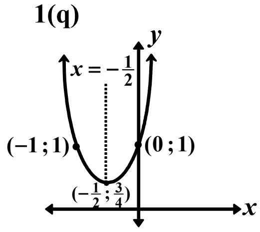
1(n)

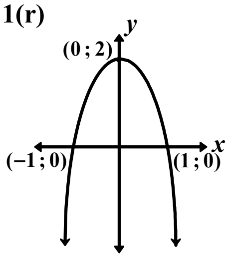


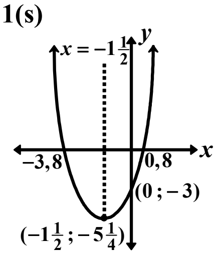
1(o)

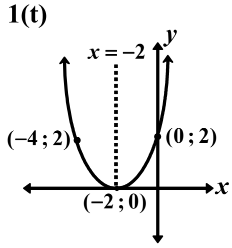


1(p)


1(q)


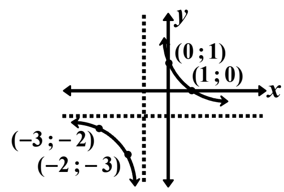
1(r)


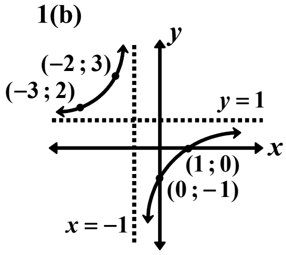
1(s)


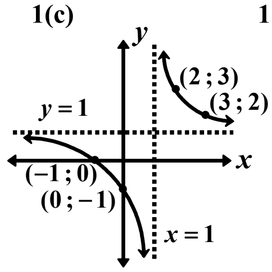
1(t)


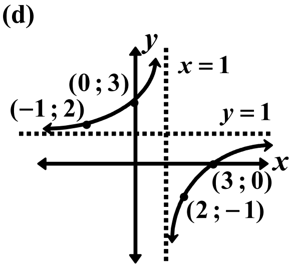
2(a) $d = 30$ **2(b)** $d = 40$ **2(c)** $t = 30$

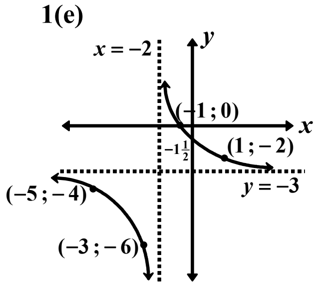
EXERCISE 5

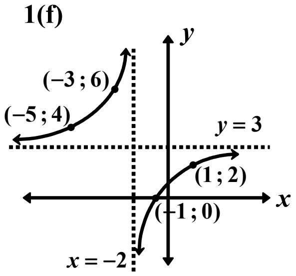
1(a)


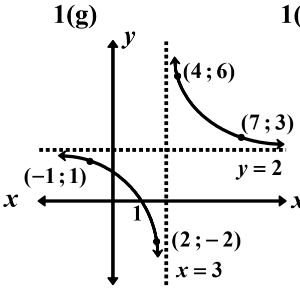
1(b)


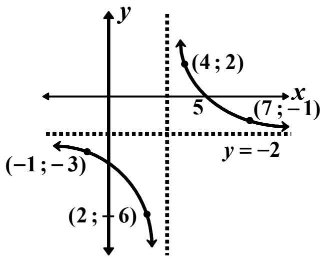
1(c)


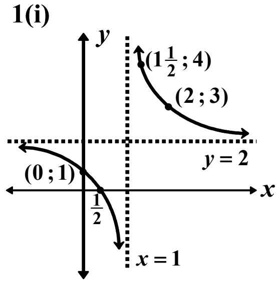
1(d)


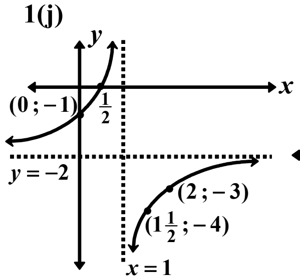
1(e)


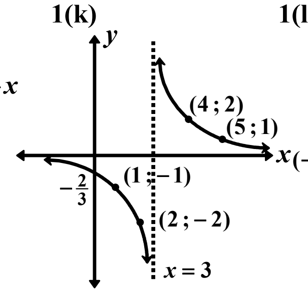
1(f)


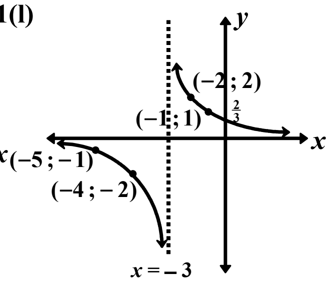
1(g)


1(h)


1(i)


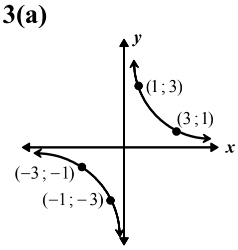
1(j)


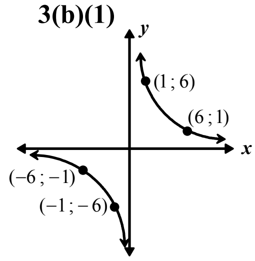
1(k)


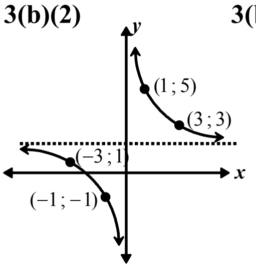
1(l)


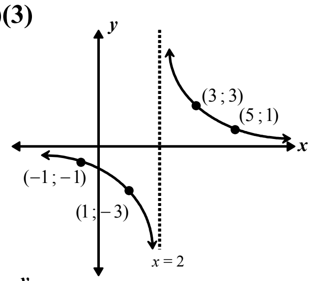
2(a) $y=\frac{2}{x}+2$ **2(b)** $y=\frac{2}{x}-2$ **2(c)** $y=\frac{2}{x+2}$ **2(d)** $y=\frac{2}{x-2}$

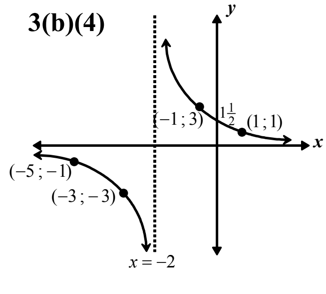
2(e) $y=\frac{2}{x+2}+3$ **2(f)** $y=\frac{2}{x-2}-3$ **2(g)** $y=-\frac{2}{x}$ **2(h)** $y=-\frac{2}{x}$

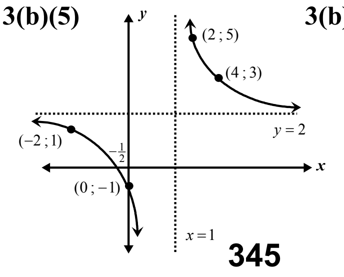
3(a)


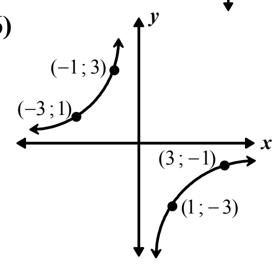
3(b)(1)


3(b)(2)


3(b)(3)


3(b)(4)


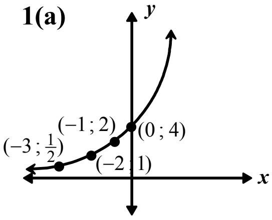
3(b)(5)


3(b)(6)


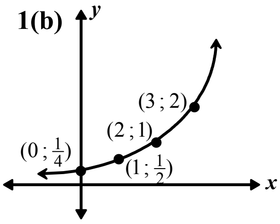
4(a) $t = 5$ **4(b)** $t = 8$

EXERCISE 6

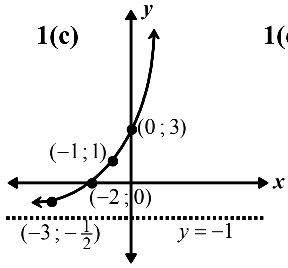
1(a)



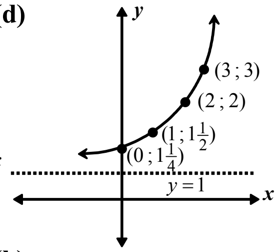
1(b)



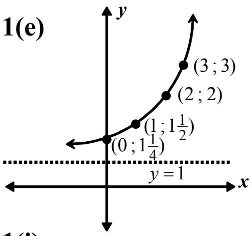
1(c)



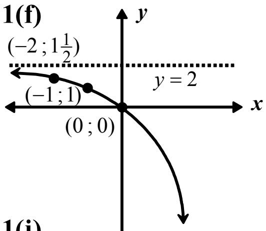
1(d)



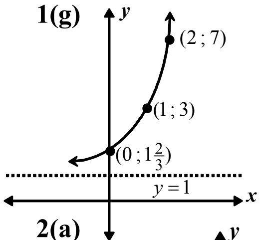
1(e)



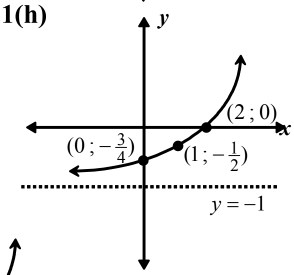
1(f)



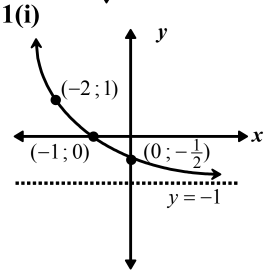
1(g)



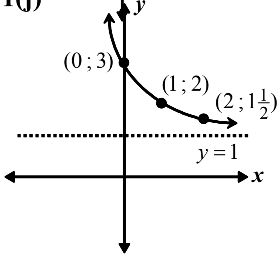
1(h)



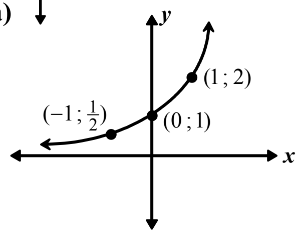
1(i)



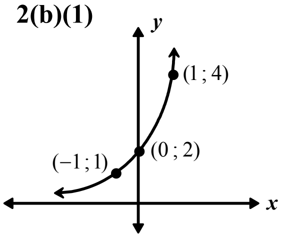
1(j)



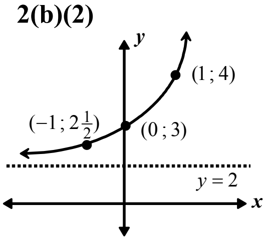
2(a)



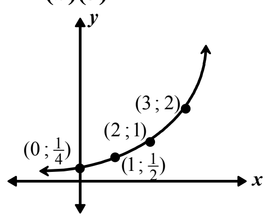
2(b)(1)



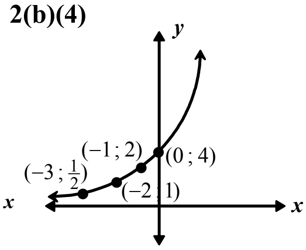
2(b)(2)



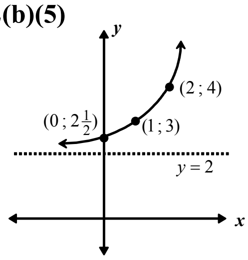
2(b)(3)



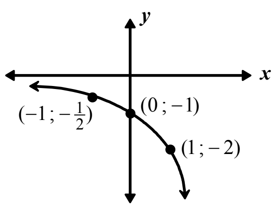
2(b)(4)



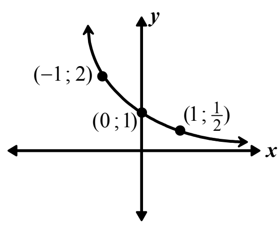
2(b)(5)



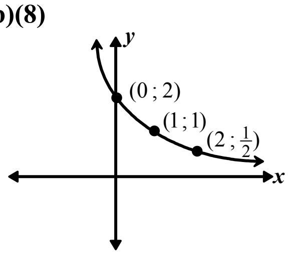
2(b)(6)



2(b)(7)



2(b)(8)



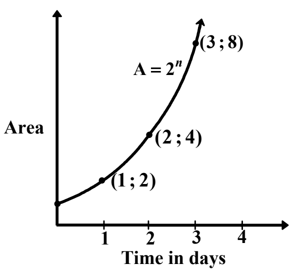
2(d) $y = -2^x$ **2(e)** $y = 2^{-x} = \left(\frac{1}{2}\right)^x$ **2(f)** $y = 2^{x-3}$ **2(g)** $y = 2^{x+3}$

2(h) $y = 2^x + 3$ **2(i)** $y = 2^x - 3$ **2(j)** $y = 2^{x-1} + 2$

3(a)

Time t	0	1	2	3	4	5	6	n
Area A	$1 = \mathbf{2^0}$	$2 = \mathbf{2^1}$	$4 = \mathbf{2^2}$	$8 = \mathbf{2^3}$	$16 = \mathbf{2^4}$	$32 = \mathbf{2^5}$	$64 = \mathbf{2^6}$	$\mathbf{2^n}$

3(b)



3(c) $A = 2^n$ **3(d)** $512\ m^2$ **3(e)** $n = 10$

3(f) $A = 2^n + 1$

EXERCISE 7

1(a) $y=-2x^2-4x+16$ **1(b)** $y=x^2+4x-5$ **1(c)** $y=\frac{5}{3}x^2+\frac{10}{3}x$

1(d) $y=2x^2+4x-3$ **1(e)** $y=-4x^2-8x+2$ **1(f)** $y=18x^2+72x+72$

1(g) $y=-x^2+16$ **1(h)** $y=-x^2+6x+7$ **2(a)** $y=\frac{3}{x}+2$

2(b) $y=\frac{2}{x-1}+1$ **2(c)** $y=\frac{-1}{x-1}-2$ **2(d)** $y=3^{x-1}+2$

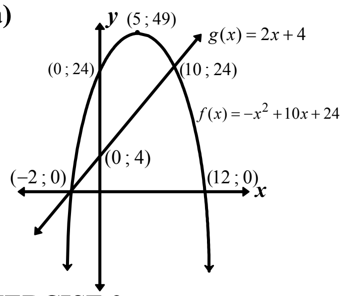
2(e) $y=\left(\frac{1}{4}\right)^{x+1}-4$ **2(f)** $y=2^{x+1}+2$

EXERCISE 8

1(a) $(-6\,;14)$ **;** $(-3\,;5)$ **1(b)** $(2\,;0)$ **;** $(-1\,;3)$ **1(c)** $(2\,;2)$

1(d) $(2\,;-3)$ **1(e)** $(1\,;4)$

2(a)



2(b) $(10\,;24)$ and $(-2\,;0)$

2(c) Domain of f: $x\in(-\infty\,;\infty)$

Domain of g: $x\in(-\infty\,;\infty)$

Range of f: $y\in(-\infty\,;49]$

Range of g: $y\in(-\infty\,;\infty)$

EXERCISE 9

1(a) $k=-1$ **1(b)** $k<-1$ **1(c)** $k>-1$ **1(d)** $-1<k<0$

1(e) $k>9$ **1(f)** $k=9$ **1(g)** $k<9$ **1(h)** $k<8$

2(b)(1) $0<k<2$ **2(b)(2)** $k>0$

EXERCISE 10

1(a) -4 **1(b)** -2 **1(c)** 2 **2(a)(1)** $1{,}25$

2(a)(2) $1{,}125$ **2(a)(3)** $1{,}1$ **2(a)(4)** $1{,}075$ **2(b)** 1 **2(c)** 1

2(d) The average gradients tend to the gradient of the tangent line to the graph of f at $x=2$

EXERCISE 11

1(a) OE $=1$ OA $=3$ **1(b)** CD $=4$ **1(c)** $(1\,;4)$ **1(d)** OF $=1$ BF $=4$

1(e) PR $=3\frac{3}{4}$ **1(f)** PT $=4$ **1(g)** OH $=4$ **1(h)** PQ $=1\frac{1}{4}$

1(i) $\frac{9}{4}$ **1(j)** $(2\,;3)$ **2(a)** OD $=4$ **2(b)** TR $=8$

2(c) $x=-2$ **2(d)** BM $=12$ **2(e)** OJ $=6$ **2(f)** FP $=10$

2(g) $\frac{25}{2}$ **3(a)** $x=1\,;y=2$ **3(b)** $(2\,;4)$ **3(c)** BC $=1$

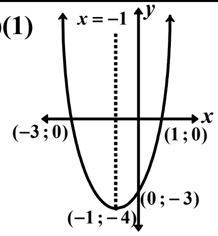
3(d) 2 **4(a)** $x<-4$ **4(b)** $x\geq-4$ **4(c)** $x=-4$ or $x=1$

4(d) $x\leq-4$ or $x\geq1$ **4(e)** $-4<x<1$ **4(f)** $x=0$ **4(g)** $x=-2$

4(h) $x\geq0$ or $x=-4$ **4(i)** $x>0$ **4(j)** $x\leq0$ **4(k)** $x<-4$ or $-4<x<0$

REVISION EXERCISE

1(a)(1)



1(a)(2) $x\in(-\infty\,;\infty)$

$y\in[-4\,;\infty)$

1(a)(3) $x>-4$

1(b)(1) $g(x)=-x^2+4x-3$

$\therefore g(x)=-(x^2-4x+3)$

$\therefore g(x)=-\left[x^2-4x+\left(\frac{-4}{2}\right)^2-\left(\frac{-4}{2}\right)^2+3\right]$

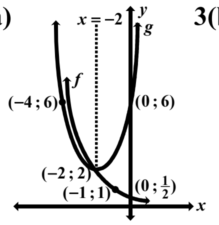
$\therefore g(x)=-\left[(x-2)^2-\left(\frac{-4}{2}\right)^2+3\right]$

$\therefore g(x)=-\left[(x-2)^2-1\right]$

$\therefore g(x)=-(x-2)^2+1$ max value $=1$

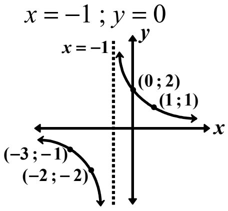
1(b)(2) $(1\,;1)$

2(a) $x=-1\,;\,y=0$ **2(c)** $f(x)=\dfrac{2}{x-2}+2$ **3(a)**



3(b) Domain of f and g: $x\in(-\infty\,;\,\infty)$; Range of f: $y\in(0\,;\,\infty)$; Range of g: $y\in[2\,;\,\infty)$

2(b)



2(d)(1) $x>-1$

2(d)(2) $-1<x\le 1$

3(c) $y=0$ **3(d)** $x=-2$ **3(e)** 2 **3(f)** $x<-2$

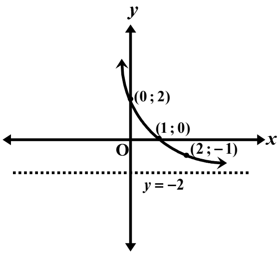
4(a) $f(x)=-(x-1)^2+4$ **4(b)** $g(x)=\dfrac{1}{x-1}+4$ **4(c)** $\left(\dfrac{3}{4}\,;\,0\right)$ **4(d)** $0\le x<\frac{3}{4}$

4(e) $x\le 0$ or $x\ge 1$ **4(f)** $-1\le x\le\frac{3}{4}$ or $1<x\le 3$ **5(a)** $f(x)=2\left(\frac{1}{2}\right)^x$

5(b) A(0 ; 2) **6(a)** $a>0\quad p<0\quad q>0$ **6(b)** $a<0\quad p>0\quad q>0$

6(c) $a<0\quad p=0\quad q<0$ **6(d)** $a>0\quad p<0\quad q>0$

5(d)



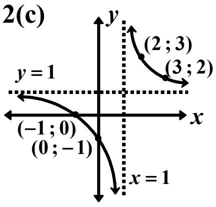
6(e) $a<0\quad p>0\quad q<0$ **6(f)** $a<0\quad p=0\quad q<0$

SOME CHALLENGES

1(a) (1 ; 1) **1(b)** $a=4$ **1(c)** $b=-\frac{7}{4}$ **2(a)** $f(x)=\dfrac{2}{x-1}+1$

2(b) $x=1$ **3(a)** C(3 ; 8) **3(b)** M$(-4\,;\,\frac{1}{16})$ **3(c)** AB = 3,6 **3(d)** $\dfrac{127}{112}$

2(c)



4(a) $k=\frac{1}{3}$ **4(b)** A(−1 ; 0) B(5 ; 0) **4(d)** D$\left(2\,;\,1\frac{4}{5}\right)$

4(e) $p<-\frac{4}{5}$ **5(a)** $f(x)=-4x+32$ **5(b)** $g(x)=-2(x-5)^2+12$

6(a) $y=-x^2+5x-4$ **6(b)** A(1 ; 0) B(4 ; 0) C(0 ; −1)

6(c) $g(x)=x-1$ **6(d)** $a=3\quad b=2$ **7(a)** $y=-\left(\frac{1}{2}\right)^x$

7(b) $y=\dfrac{8}{x}$ **7(c)** one solution $x=-2$

CHAPTER 6

REVISION EXERCISE

1(a) $\dfrac{2}{\sqrt{5}}$ **1(b)** $\dfrac{4}{5}$ **2(a)** $-\dfrac{5}{12}$ **2(b)** $\dfrac{49}{169}$ **3(a)** $\dfrac{4}{3}$ **3(b)** -17

4. $6\sqrt{6}$ **5.** $-\dfrac{73}{60}$

EXERCISE 1

1(a) $-\sin\theta$ **1(b)** $\sin\theta$ **1(c)** $-\cos\theta$ **1(d)** $\cos\theta$ **1(e)** $-\tan\theta$

1(f) $\tan\theta$ **1(g)** $\cos^2\theta$ **1(h)** $\tan^2\theta$ **1(i)** $-\sin^2\theta$ **2(a)** -1

2(b) 0 **2(c)** $\dfrac{\tan\theta}{2}$ **2(d)** -1

EXERCISE 2

1(a) $\cos\theta$ **1(b)** $\sin\theta$ **1(c)** $-\sin\theta$ **1(d)** $-\cos\theta$ **1(e)** $\cos\theta$

1(f) $\cos\theta$ **1(g)** $-\sin\theta$ **1(h)** $\sin\theta$ **1(i)** $\sin^2\theta$ **2(a)** -1

2(a) -1 **2(b)** $-\cos^2\theta$ **2(c)** 0 **3(a)** 0 **3(b)** -1

3(c) $-\sin\theta$ **3(d)** $-\cos\theta$ **3(e)** $\sin\theta$

EXERCISE 3

(a) $-\sin 50^\circ$ **(b)** $-\tan 70^\circ$ **(c)** $-\cos 85^\circ$ **(d)** $\tan 55^\circ$ **(e)** $-\sin 70^\circ$

(f) $-\cos 42°$ **(g)** $\cos 25°$ **(h)** $\sin 60°$ **(i)** $\cos 56°$

EXERCISE 4

1(a) $-\tan\theta$ **1(b)** $\cos\theta$ **1(c)** $-\sin\theta$ **1(d)** $-\sin\theta$ **1(e)** $-\tan\theta$
1(f) $\sin\theta$ **1(g)** $\cos\theta$ **1(h)** $\cos\theta$ **1(i)** $-\tan\theta$ **1(j)** $-\cos\theta$
1(k) $-\cos\theta$ **1(l)** $\sin\theta$ **2(a)** -1 **2(b)** -1 **2(c)** $-\tan\beta$
3(a) $-\tan 50°$ **3(b)** $-\cos 60°$ **3(c)** $-\sin 20°$ **3(d)** $-\tan 30°$ **3(e)** $-\cos 45°$
3(f) $-\sin 55°$ **3(g)** $-\tan 70°$ **3(h)** $\cos 45°$ **3(i)** $\sin 55°$ **3(j)** $-\sin 55°$

EXERCISE 5

1(a) p **1(b)** $-p$ **1(c)** $-p$ **1(d)** p **1(e)** $\sqrt{1-p^2}$
1(f) $\frac{p}{\sqrt{1-p^2}}$ **2(a)** $-k$ **2(b)** k **2(c)** k **2(d)** $-k$
2(e) $-\sqrt{1-k^2}$ **3(a)** t **3(b)** $-t$ **3(c)** $-t$ **3(d)** $-t$ **3(e)** $\frac{1}{\sqrt{1+t^2}}$

EXERCISE 6

1(a) $\frac{3}{4}$ **1(b)** -1 **1(c)** $-2\frac{1}{2}$ **1(d)** 0 **1(e)** $\frac{2}{\sqrt{3}}$
1(f) 1 **1(g)** 2 **1(h)** 1 **1(i)** $\frac{2}{3}$ **1(j)** -1
1(k) -1 **1(l)** $\frac{1}{2}$ **2.** -1 **3.** 4 **5.** $30°$

EXERCISE 7

(a) $\tan\theta$ **(b)** $\sin\theta$ **(c)** $\cos\theta$ **(d)** $\frac{1}{\cos^2\theta}$ **(e)** $-\tan^2\theta$
(f) $\cos^2\theta$

EXERCISE 9

(a) $\cos^2\theta$ **(b)** -1 **(c)** $\sin\theta$ **(d)** 1 **(e)** 1 **(f)** -1

EXERCISE 10

1(a) $63{,}43°$; $243{,}43°$ **1(b)** $48{,}93°$; $311{,}07°$ **1(c)** $214{,}06°$; $325{,}94°$
1(d) $131{,}81°$; $228{,}19°$ **1(e)** $48{,}59°$; $131{,}41°$ **1(f)** $22{,}29°$
2. $231{,}38°$; $308{,}62°$

EXERCISE 11

1. $33{,}43°$; $231{,}43°$ **2.** $21{,}09°$; $98{,}91°$ **3.** $98{,}13°$; $171{,}87°$ **4.** $82{,}30°$; $123{,}70°$

EXERCISE 12

1(a) $111{,}80° + k180°$; $291{,}80° + k180°$ **1(b)** $64{,}42° + k360°$; $293{,}58° + k360°$
1(c) $86{,}89° + k360°$; $353{,}11° + k360°$ **1(d)** $122{,}07° + k360°$; $217{,}93° + k360°$
1(e) $15{,}73° + k180°$; $74{,}27° + k180°$ **1(f)** $23{,}86 + k60°$; $83{,}86° + k60°$
2(a)(i) $111{,}04° + k180°$; $291{,}04° + k180°$ **2(a)(ii)** $\theta \in \{-68{,}96°;111{,}04°;291{,}04°\}$
2(b)(i) $109{,}27° + k360°$; $250{,}73° + k360°$ **2(b)(ii)** $\theta \in \{-109{,}27°;109{,}27°;250{,}73°\}$
2(c)(i) $111{,}18° + k720°$; $248{,}82° + k720°$ **2(c)(ii)** $\theta \in \{111{,}18°;248{,}82°\}$
3. $\theta \in \{161{,}54°;318{,}46°;521{,}54°;678{,}46°\}$ **4.** $\theta \in \{-44{,}63°;18{,}63°\}$
5. $\theta \in \{-87{,}86°;2{,}14°;92{,}14°\}$

EXERCISE 13

1(a) $30° + k180°$; $210° + k180°$ **1(b)** $90° + k360°$; $270° + k360°$; $60° + k360°$; $300° + k360°$
1(c) $135° + k180°$; $315° + k180°$; $30° + k360°$; $150° + k360°$
1(d) $135° + k360°$; $330° + k360°$; $90° + k360°$ **2.** $\theta \in \{53{,}13°;233{,}13°\}$ **3.** $\theta \in \{-60°;60°\}$
4. $\theta \in \{63{,}43°;243{,}43°\}$ **5.** $108{,}43° + k180°$; $288{,}43° + k180°$

EXERCISE 14

1(a) $90° + k180°$ **1(b)** $135° + k180°$; $315° + k180°$ **1(c)** $90° + k180°$ **2.** $\theta \in \{0°;180°\}$

EXERCISE 15

1(a) $\theta \in \{120°;240°\}$ **1(b)** $\theta \in \{60°;120°\}$ **1(c)** $\theta \in \{120°;300°\}$

1(d) $\theta \in \{-150°;30°\}$ **1(e)** $\theta \in \{-210°;30°;150°\}$ **2.** $27° + k360°$ **3.** $90° + k360°$; $270° + k360°$

4(a) $\theta \in \{0°;30°;150°;360°\}$ **4(b)** $\theta \in \{0°;120°;180°;240°;360°\}$ **5.** $\theta \in \{135°;315°\}$

6(b)(i) $\theta = k.180°$ **6(b)(ii)** $\theta = 90° + k.180°$ **6(b)(iii)** $\theta = k.360°$ **6(b)(iv)** $\theta = 90° + k.360°$

EXERCISE 16

1. 11,32 **2.** 4 **3.** 37,87° **4.** 40,54°

EXERCISE 17

1(a) $\theta = 46,29°$; $\beta = 63,71°$; $x = 12,40$ **1(b)** $\theta = 44°$; $x = 17,07$; $y = 12,98$

1(c) $\theta = 53,13°$ **1(d)** $x = 31,57$ **1(e)** $x = 79,81$; $y = 39,91$ **2.** 12,11

3. 124,53° **4(a)** 7,78 **4(b)** 5,96

EXERCISE 18

1(a) 10,09 **1(b)** 106,07° **1(c)** 48,35° **1(d)** 23,89 **2(a)** 25,33° **2(b)** 83,71°

EXERCISE 19

1(a) 25,4 **1(b)** 15,16 **1(c)** 1090,43 **1(d)** 94,62 **2(a)** 2820,12 **2(b)** 101,44 **3.** $19,4km^2$

EXERCISE 20

1. $P(\cos\theta;\sin\theta)$ **2.** $p\tan\theta$ **5(a)** $\dfrac{h}{\sin\alpha}$ **5(b)** $\alpha - \beta$

5(d) 197,51 **6(a)** $\dfrac{1}{2}r^2\sin 72°$ **6(b)** $\dfrac{5}{2}r^2\sin 72°$ **6(d)** $172,05\,\text{cm}^2$

REVISION EXERCISE

1(a) $-2\sqrt{6}$ **1(b)** $-\dfrac{1}{5}$ **2(a)** 0,6 **2(b)** 0,3 **3(a)** -1 **3(b)** 1

3(c) $\dfrac{2+3\sqrt{3}}{6}$ **3(d)** 1 **5(a)** -1 **5(b)** $-\sin^2\theta$ **5(c)** $\tan^2\theta$

6(a) $35,82° + k120°$; $84,18° + k120°$ **6(b)** $185,74° + k360°$; $354,26° + k360°$

6(c) $154,74° + k180°$; $334,74° + k180°$ **6(d)** $90° + k180°$ **6(e)** $k180°$; $k360°$ **6(f)** $90° + k180°$

7(b) $\theta \in \{-341,57°;-251,57°;-161,57°\}$ **8(a)** $-m$ **8(b)** $-m$ **8(c)** $\sqrt{1-m^2}$

8(d) $\dfrac{m}{\sqrt{1-m^2}}$ **9(b)** $180° + k360°$

SOME CHALLENGES

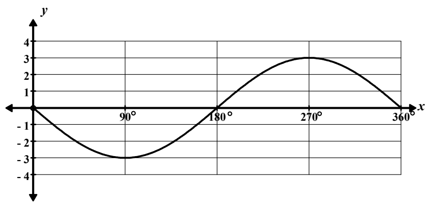
1. -1 **2.** $\cos\theta$ **3.** $-0,624$ **4(a)** $\dfrac{5}{13}$ **4(b)** 7,2 ; $(-7,2\,;3)$ **5.** 0,5

7. $44\dfrac{1}{2}$ **8.** 7,15 **9(a)** 3,35 **9(b)** 63,43°

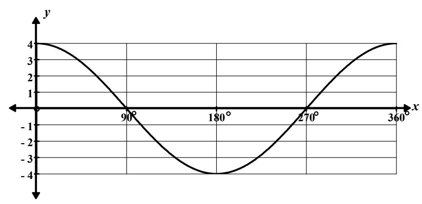
TRIGONOMETRIC FUNCTIONS

EXERCISE 1

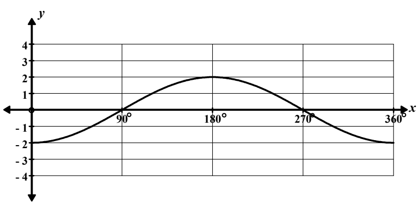
1(a)



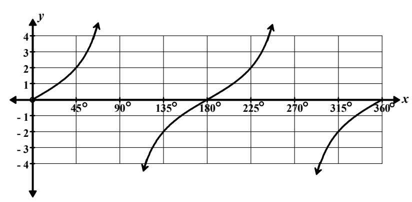
1(b)



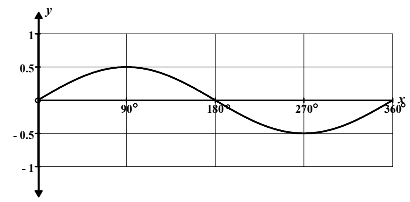
1(c)



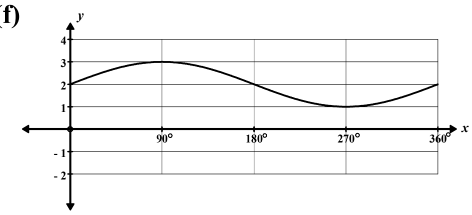
1(d)



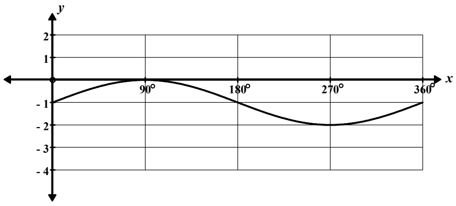
1(e)



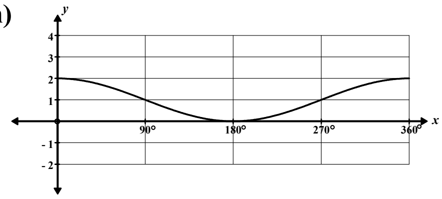
1(f)



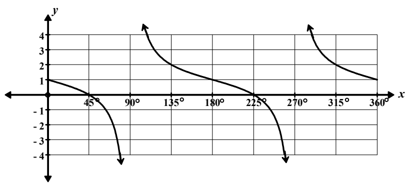
1(g)



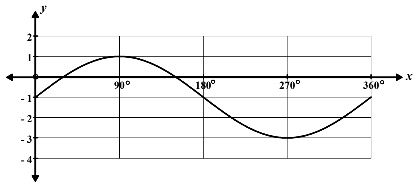
1(h)



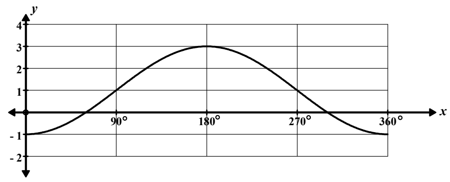
1(i)



1(j)



1(k)



2(a) $f(x) = -2\sin x + 2$; $g(x) = 3\cos x$

2(b) Max f: 4 Min f: 0
Max g: 3 Min g: -3

2(c) Amplitude f: 2 Amplitude g: 3

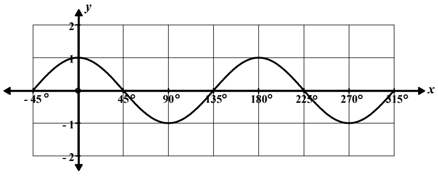
2(d) 5

2(e)(1) $0° \le x \le 90°$ or $x = 270°$ **2(e)(2)** $90° < x < 270°$ **2(e)(3)** $0° \le x < 90°$

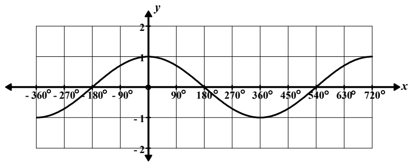
2(e)(4) $90°$ **2(e)(5)** $90° \le x < 270°$ **2(e)(6)** $90°$ **2(e)(7)** $180°$ **2(e)(7)** $270°$

<u>EXERCISE 2</u>

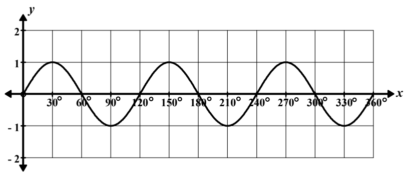
(a)



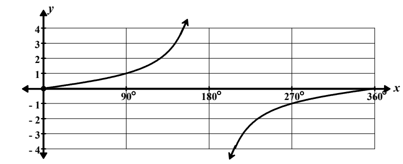
(b)



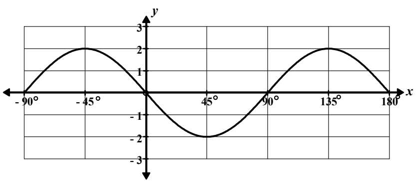
(c)



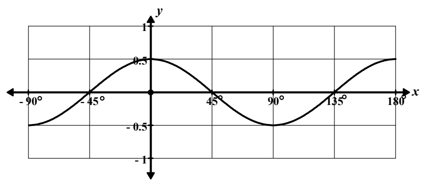
(d)



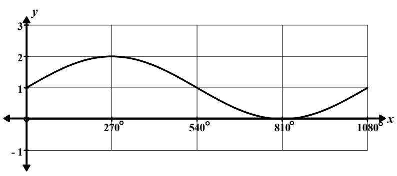
(e)



(f)

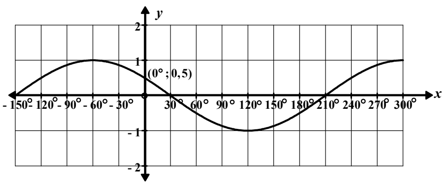


(g)

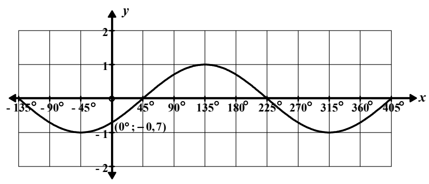


<u>EXERCISE 3</u>

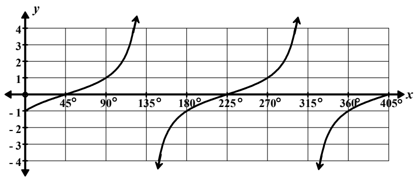
(a)



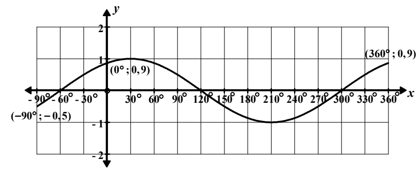
(b)



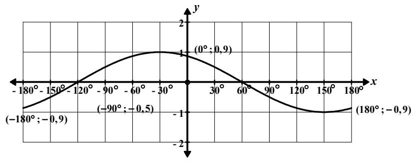
(c)



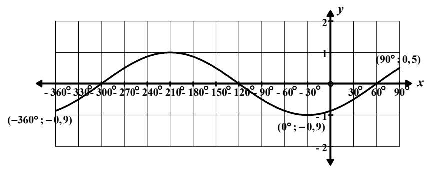
(d)



(e)

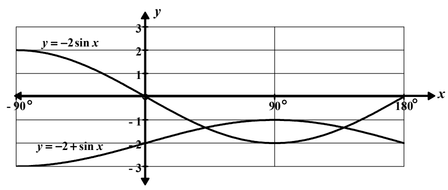


(f)

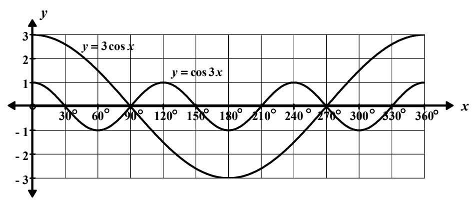


REVISION EXERCISE

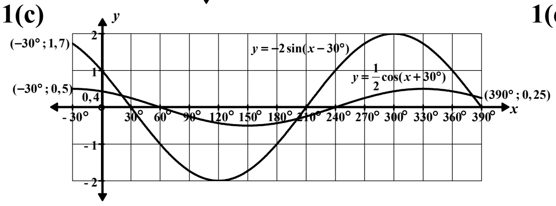
1(a)



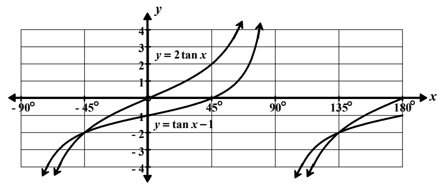
1(b)



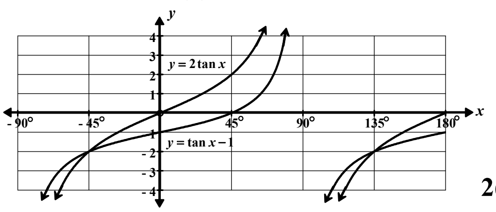
1(c)

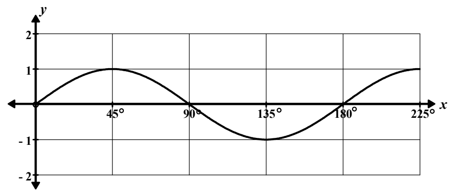


1(d)



2(a) **2(b)** (3) **2(c)**





2(d) $y = \sin 2x$

3(a) $f(x) = \sin(x - 30°)$ **3(b)** $g(x) = \cos 3x$

3(c)(1) $x = -60°$ or $x = 30°$ or $x = 120°$ or $x = 210°$ **3(c)(2)** $-60° < x < 30°$

3(c)(3) $30° < x < 120°$ or $120° < x < 210°$ **3(c)(4)** $30° \leq x \leq 210°$

3(c)(5) $-30° \leq x \leq 90°$ or $150° \leq x \leq 210°$

3(c)(6) $-60° \leq x \leq -30°$ or $90° \leq x \leq 150°$ or $x = 30°$ or $x = 210°$

3(c)(7) $x = 30°$ or $x = 210°$ **3(c)(8)** $30° \leq x \leq 210°$

3(c)(9) $x = -30°$ or $x = 30°$ or $x = 90°$ or $x = 150°$ or $x = 210°$

3(c)(10) $-60° \leq x < -30°$ or $30° < x < 90°$ or $150° < x < 210°$ **3(d)** 720°

4(a) $f(x) = \sin x + 2$ $g(x) = -\sin 2x$ $h(x) = \cos 2x$ $j(x) = -\tan x$ **4(b)** 1

4(c) f : 360° g :180° **4(d)(1)** $-90° < x < 0°$ or $90° < x < 180°$

4(d)(2) $-90° < x < 90°$ or $90° < x < 180°$ **4(d)(3)** $x = 0°$ or $x = 180°$

4(d)(4) $-45° \leq x \leq 0°$ or $45° \leq x < 90°$ or $135° \leq x \leq 180°$

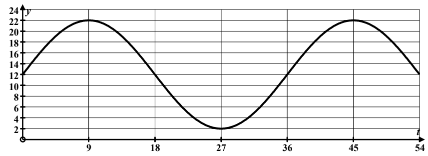
5. $f(x) = \sin(x - 60°)$ $g(x) = \sin(x + 60°)$ or $g(x) = \cos(x - 30°)$

SOME CHALLENGES

1(a) $10m$ **1(b)** 11,1 hours **1(c)** 15,9 hours **2(a)** $0,5m$

2(b) A(0°;0,5) ; B(120°;0,5) ; C(360°;0,5) ; D(480°;0,5) ; E(720°;0,5)

2(c) $1,5m$ **3(a)** $a = 12$ **3(b)** $b = 10$ **3(c)** 3, 15, 39, 51 **3(d)**



CHAPTER 7

REVISION EXERCISE

1. $1,05\text{cm}^2$ **2(a)** $(1600\pi)\,cm^3$ **2(b)** $(420\pi)\,cm^2$ **2(c)** doubled **3(a)** 30cm^3

3(b) 82cm^2 **4.** $273,3\text{cm}^3$ **5.** $7079,1\text{cm}^3$; $1841,9\text{ cm}^2$ **6(a)** $12,5m$

6(b) $93,75m^2$ **6(c)** $1800m^2$ **6(d)** $5250m^3$

CHALLENGE

$2036,9\text{cm}^3$; $1225,7\text{cm}^2$

CHAPTER 8

EXERCISE 1

(a) 16 **(b)** 1 **(c)(1)** 25 **(c)(2)** $20\sqrt{6}$ **(d)(1)** $x+8$ **(d)(2)** 13
(e) 20

EXERCISE 2

(a) 30° **(b)** 48° **(c)** 20° **(d)** 18° **(e)** 240° **(f)** 115°
(g) 105° **(h)** $x = 47,5°$; $y = 132,5°$ **(i)** $x = 40°$; $y = 80°$ **(j)** 10°

EXERCISE 3

(a) 35° **(b)** 38° **(c)** $x = 38°$; $y = 52°$ **(d)** $x = 60°$; $a = 120°$; $y = 68°$

EXERCISE 4

(a) $x = 18°$; $y = 22°$ **(b)** $x = 16°$; $y = 16°$ **(c)** 35° **(d)** $x = 32°$; $y = 56°$
(e) 35° **(f)** $x = 44°$; $y = 46°$; $z = 46°$ **(g)** $x = 32°$; $y = 75°$; $z = 75°$
(h) $a = 20°$; $b = 50°$; $c = 50°$; $d = 40°$; $e = 80°$

EXERCISE 5

1(a) 15° **1(b)** 72,5° **2.** 60°

EXERCISE 6

(a) $x = 97°$; $y = 75°$ **(b)** $x = 98°$; $y = 58°$; $z = 20°$ **(c)** $x = 35°$; $y = 70°$; $m = 55°$
(d) 80° **(e)** $x = 100°$; $y = 45°$ **(f)** $x = 36°$; $y = 60°$ **(g)** 42°
(h) $x = 130°$; $y = 90°$; $z = 40°$ **(i)** $x = 105°$; $y = 30°$

EXERCISE 7

1(a) 110° **1(b)** 85° **1(c)** $x = 125°$; $y = 90°$; $z = 35°$ **1(d)** $a = 112°$; $b = 88°$
1(e) $x = 98°$; $y = 90°$ **1(f)** $x = 106°$; $m = 74°$; $y = 106°$ **1(g)** $x = 30°$; $y = 50°$

EXERCISE 8

(a) 55° **(b)** $a = 18°$ **(c)** $x = 40°$; $y = 10°$ **(d)** $x = 50°$; $m = 40°$
(e) $a = 52°$; $b = 38°$; $c = 38°$ **(f)** $x = 30°$; $y = 60°$ **(g)** $x = 70°$; $m = 20°$; $y = 70°$

EXERCISE 9

(a) $a = 65°$; $b = 130°$; $c = 65°$ **(b)** $x = 30°$; $y = 70°$

EXERCISE 10

1(a) $x = 60°$; $y = 70°$ **1(b)** $x = 100°$; $y = 40°$ **1(c)** $x = 30°$; $y = 70°$ **1(d)** 80°
1(e) $x = 35°$; $y = 35°$ **1(f)** $x = 9°$ **1(g)** $x = 30°$; $y = 70°$ **1(h)** $x = 26°$; $y = 26°$
1(i) $x = 44°$; $y = 112°$ **1(j)** $x = 96°$; $y = 84°$ **1(k)** $x = 40°$; $y = 31°$
1(l) $x = 130°$; $y = 50°$; $z = 50°$ **1(m)** $x = 54°$; $y = 92°$ **2(b)** $x = 16°$

EXERCISE 11

1. 6cm **2(a)** 13° **2(b)** 154° **3(a)** 52° **3(b)** 32° **3(c)** 38° **3(d)** 20°
4(a) 40° **4(b)** 40° **4(c)** 75° **4(d)** 15° **5(a)** 110° **5(b)** 70° **5(c)** 100°
5(d) 10° **7(a)** 90° **7(b)** 55° **7(c)** 35° **7(d)** 20° **7(e)** 20°

REVISION EXERCISE

1(a) 22° **1(b)** 22° **1(c)** 68° **1(d)** 22° **1(e)** 44° **1(f)** 46° **2(b)** 12
3. $x = 30°$ **5(b)** $90° - 2x$

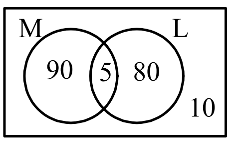
SOME CHALLENGES

1(b) 24,1

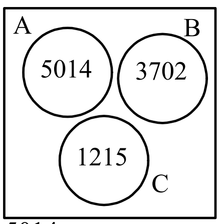
CHAPTER 9

EXERCISE 1

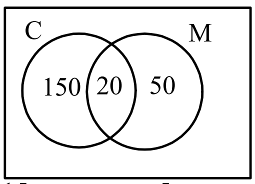
1(a)



2(a)



3(a)



1(b) $\frac{18}{37}$ **1(c)** $\frac{1}{37}$ **2(b)** $\frac{5014}{9931}$ **3(b)** $\frac{15}{22}$ **3(c)** $\frac{5}{22}$

1(d) $\frac{2}{37}$ **1(e)**Inclusive **2(c)**Mutually exclusive **3(d)** Inclusive

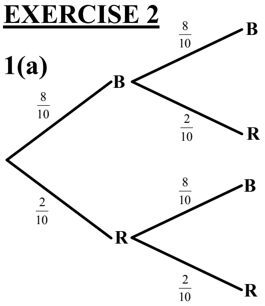
1(f)not complementary **2(d)**Complementary **3(e)**Not complementary

4(a)S and T **4(b)** T and H; S and H **4(c)**None **4(d)** $\frac{5}{6}$

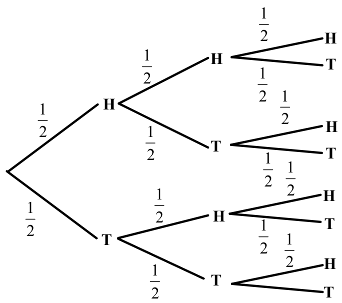
5(a) 0,125 **5(b)(1)**not mutually exclusive **5(b)(2)**Inclusive **5(b)(3)**not comp

EXERCISE 2

1(a)



2(a)



3(b) $\frac{12}{49}$ **3(c)** $\frac{12}{49}$ **3(d)** $\frac{16}{49}$ **3(e)** $\frac{24}{49}$

4(b) $\frac{48}{343}$ **4(c)** $\frac{27}{343}$ **4(d)** $\frac{108}{343}$

1(b) $\frac{1}{25}$ **1(c)** $\frac{8}{25}$ **2(b)** $\frac{1}{8}$ **2(c)** $\frac{1}{4}$ **2(d)** $\frac{7}{8}$ **2(e)** $\frac{1}{4}$

5(b) $\frac{1}{36}$ **5(c)** $\frac{2}{36}$ **5(d)** $\frac{10}{36}$ **5(e)** $\frac{35}{36}$

EXERCISE 3

1(a) $\frac{2}{11}$ **1(b)** $\frac{3}{11}$ **1(c)** $\frac{6}{11}$ **1(d)** $\frac{9}{11}$ **2(b)** $\frac{242}{250}\times\frac{241}{249}\times\frac{240}{248}$

2(c) $\frac{8}{250}\times\frac{242}{249}\times\frac{241}{248}+\frac{242}{250}\times\frac{8}{249}\times\frac{241}{248}+\frac{242}{250}\times\frac{241}{249}\times\frac{8}{248}$

2(d) $\frac{8}{250}\times\frac{7}{249}\times\frac{242}{248}+\frac{8}{250}\times\frac{242}{249}\times\frac{7}{248}+\frac{242}{250}\times\frac{8}{249}\times\frac{7}{248}$

2(e) $\frac{8}{250}\times\frac{7}{249}\times\frac{6}{248}$ **3(a)** $\frac{1}{15}$ **3(b)** $\frac{4}{15}$ **3(c)** $\frac{11}{15}$ **4.** 0,61

5(b) $\frac{3}{7}\times\frac{4}{6}\times\frac{3}{5}$ **5(c)** $\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}$ **5(d)** $\frac{3}{7}\times\frac{2}{6}\times\frac{4}{5}+\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}+\frac{3}{7}\times\frac{4}{6}\times\frac{2}{5}$

EXERCISE 4

1(a) $\frac{3}{8}$ **1(b)** $\frac{5}{8}$ **1(c)** $\frac{5}{8}$ **1(d)** $\frac{3}{8}$ **1(e)** $\{1,2,3,4,5,6\}$ **1(f)** $\frac{6}{8}$

1(g) $\{4,5\}$ **1(h)** $\frac{2}{8}$ **1(i)** $\{7,8\}$ **1(j)** $\frac{2}{8}$ **1(k)** $\{1,2,3,6,7,8\}$ **1(l)** $\frac{6}{8}$

1(m) $\{6\}$ **1(n)** $\frac{1}{8}$ **1(o)** $\{1,2,3\}$ **1(p)** $\frac{3}{8}$ **1(q)** $\{4,5,6,7,8\}$ **1(r)** $\frac{5}{8}$

1(s) $\{1,2,3,4,5,7,8\}$ **1(t)** $\frac{7}{8}$ **1(u)** 1 **1(v)** 0 **1(w)** 1 **1(x)** $\frac{3}{5}$

2(a) 0,1 **2(b)** 0,1 **2(c)** 0,2 **2(d)** 0,3

2(e) 0,2 **2(f)** 0,9 **3(a)** 0,9 **3(b)** 0,55 **3(c)** 0,9 **3(d)** 0,75 **3(e)** 0,1

4(b)(1) $\frac{50}{130}$ **4(b)(2)** $\frac{32}{130}$ **5(b)** $\frac{16}{30}$ **5(c)** $\frac{6}{30}$ **5(d)** $\frac{17}{30}$ **5(e)** $\frac{9}{30}$

6(b) 33 **6(c)** $\frac{16}{80}$ **7(b)** 12 **7(c)** 10 **7(d)** 12 **7(e)** $\frac{55}{200}$

EXERCISE 5

1(a) $\frac{25}{100}$ **1(b)** $\frac{65}{100}$ **1(c)** $\frac{7}{65}$ **1(d)** $\frac{25}{35}$ **1(e)** $\frac{35}{100}$ **1(f)** $\frac{25}{65}$ **2(a)** $\frac{4}{15}$

2(b) $\frac{4}{12}$ **2(c)** $\frac{4}{18}$ **2(d)** $\frac{6}{10}$ **3(a)(1)** $\frac{61}{151}$ **3(a)(2)** $\frac{137}{151}$ **3(a)(3)** $\frac{5}{151}$ **3(a)(4)** $\frac{5}{14}$

3(a)(5) $\frac{9}{10}$ **3(b)** Not independent

REVISION QUESTIONS

1(a) $\frac{15}{77}$ **1(b)** $\frac{40}{77}$ **2(a)** 8 **2(b)** 12 **2(c)** 8 **2(d)** $\frac{1}{2}$

3(b)(1) 0,382347 **3(b)(2)** 0,867349 **3(b)(3)** 0,7497 **4(b)**Independent

5(b)(1) Inclusive **5(b)(2)**Dependent

SOME CHALLENGES

1(b)(1) $\frac{1}{10}$ **1(b)(2)** $\frac{31}{40}$ **1(b)(3)** $\frac{29}{40}$ **2(a)** $\frac{11}{36}$ **2(b)** $\frac{10}{36}$ **2(c)** $\frac{26}{36}$ **2(d)** $\frac{4}{10}$

2(e) $\frac{1}{6}$ **2(f)** $\frac{7}{11}$

CHAPTER 10

EXERCISE 1

1(a) R8600 **1(b)** R9869,11 **2(a)** R7853,40 **2(b)** R6375,91 **3.** R177 020,93

4. R13008,69 **5.** R29 411,76 **6.** $r = 16,7\%$ **7.** R227793,27 **8.** $r = 3,9\%$

EXERCISE 2

1(a) R54 400 **1(b)** R79 517 **2(a)** R3428,57 **2(b)** R2407,60 **3(a)** $r = 10\%$

3(b) $r = 18,2\%$ **4.** R12 754,37 **5.** R1600 **6.** R44 260,32 **7.** $r = 12,5\%$

8(a) $r = 25\%$ **8(b)** R750 000 ; R500 000 ; R250 000 ; R0

EXERCISE 3

1(a) R46261,22 **1(b)** R47635,59 **1(c)** R48388,76 **1(d)** R48918,41 **1(e)** R49182,97

2. R17 757,12 **3.** Annual: R328413,95 Monthly: R359748,12 Monthly is the better option

4. R521 877,24 **5.** Option B **6.** R410796 7. R80545,25 **8.** R178880,20

EXERCISE 4

1(a) $r = 13,4\%$ **1(b)** $r = 15,9\%$ **1(c)** $r = 11,6\%$ **1(d)** $r = 15\%$ **2(a)** $r = 12,6\%$

2(b) $r = 14\%$ **2(c)** $r = 10\%$ **3(a)** R125 126,14 **3(b)** 14,8% **3(c)** R125 126,14

4(a) $r = 4,5\%$ **4(b)** $i = 0,04380714441$ **5.** Bank A

EXERCISE 5

1. R41 996,50 **2.** R82 126 **3.** R354 924,13

4(a) $i = 0,08299950681$ **;** $i = 0,1025$ **4(b)** R17078,20 **5.** R6436,77 **6.** R2767217,60

7. R35002,50 **8(a)** R58844,11 **8(b)** $i = 0,1493420292$ **;** $i = 0,134225$ **8(c)** R58844,11

EXERCISE 6

1. R19 779 **2.** R40 282,08 **3.** R28077,04 **4.** R14759,27 **5.** R21737,74

6. R151 735,36 **7.** R3587,42 **8.** R15914,71 **9.** R58590,88 **10.** R20916,83

11. R280980,71

REVISION EXERCISE

1(a) R32303,17 **;**Yes **1(b)** R200184,81 **1(c)** R132893,44 **1(d)** R258944,73

2(a) R10406,73 **2(b)** 0,08299950681 **2(c)** R10406,73 **2(d)** R15868,74

2(e) R11664 **3.** R124 795,90 **4(a)** R32270,04 **4(b)** R26767,38

4(c) R36080,75 **5.** R7403,37

SOME CHALLENGES

1. R29231,67 **2.** R6770,86 **3.** R61651,50 **4.** R6000

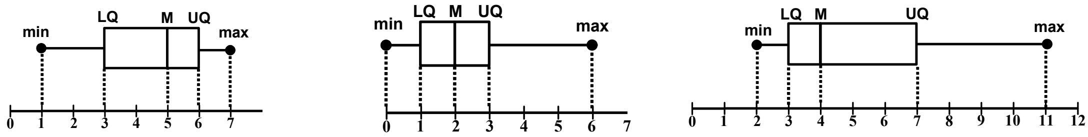
CHAPTER 11

EXERCISE 1

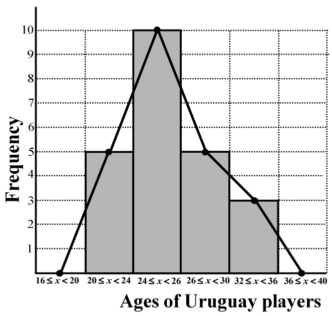
1(a) 4,6 **1(b)** 8 **1(c)** 5 **1(d)** 1 **1(e)** 3; 5; 6 **1(f)** 4; 8; 11

1(g) 3 **1(h)** 7 **1(i)** 2 **1(j)** 1 ; 2 **1(k)** 3 ; 4 **1(l)** 1

1(m) matches played wins goals against



2(b)



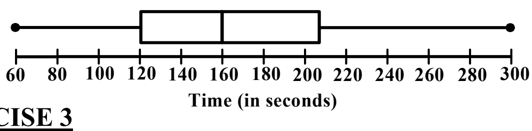
2(c)	26,8	**2(d)**	27	**2(e)**	23
2(f)	26	**2(g)**	12	**2(h)**	24 ; 29
2(i)	2,5				
3(a)	11,5	**3(b)**	12,5 `	**3(c)**	$5 \le x < 10$
4(b)	34,4				

EXERCISE 2

1(c) $Q_1 \approx 44$ **1(d)** $Q_2 \approx 55$ **1(e)** $Q_3 \approx 55$ **2(c)** $Q_1 \approx 19,5$

2(d) $Q_2 \approx 23$ **2(e)** $Q_3 \approx 29$ **3(c)** $48 \le x \le 49$ **4(b)** 160 **4(c)** 9

4(d)



4(e) 164

5(c) $570 \le t \le 580$

5(d) $650 \le t \le 900$

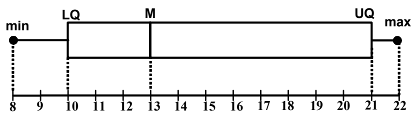
EXERCISE 3

1(a) A: $\bar{x} = 6,1$ B: $\bar{x} = 7,3$ C: $\bar{x} = 4,7$ D: $\bar{x} = 6$

A: SD = 3,5 B: SD = 2,8 C: SD = 3,0 D: SD = 3,1

2(a) 2,25 **3(a)** 3,47706773 **3(b)** 12,1 **4(a)** 11

4(b)



4(c) 11 **4(d)** 5,215361924

4(e) 27,2

EXERCISE 4

1(a) A: $\bar{x} = 6,1$ B: $\bar{x} = 7,3$ C: $\bar{x} = 4,7$ D: $\bar{x} = 6$

1(b)

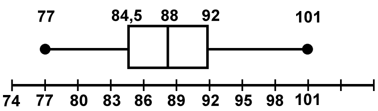
Learner A:	Min: 1	$Q_1 = 1,5$	$Q_2 = 7$	$Q_3 = 9$	Max: 10
Learner B:	Min: 1	$Q_1 = 7$	$Q_2 = 8$	$Q_3 = 9$	Max: 10
Learner C:	Min: 1	$Q_1 = 2$	$Q_2 = 4$	$Q_3 = 8$	Max: 10
Learner D:	Min: 2	$Q_1 = 3$	$Q_2 = 6$	$Q_3 = 9$	Max: 10

1(d) Learner A: mean < median Learner B: mean < median

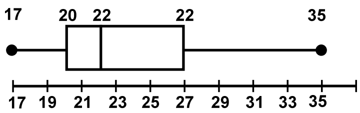
Learner C: mean > median Learner D: mean = median

2(a) 88 for all three **2(b)** $Q_1 = 84,5$ $Q_3 = 92$ **2(c)** 5,8 **2(d)** 67%; 96%; 100%

2(g)



3(d)



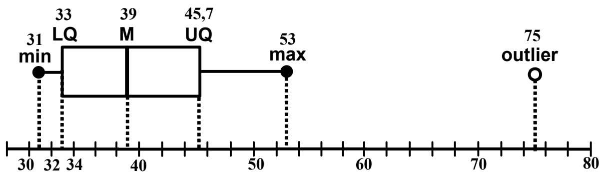
3(b) positively skewed, mean > median

EXERCISE 5

1(a) $Q_1 = 33$ $M = Q_2 = 39$ $Q_3 = 47,5$ $IQR = 14,5$

Interval for non-outliers $= [11,25 ; 69,25]$ Outlier is Sportsman 1 with 75 million dollars

1(b)

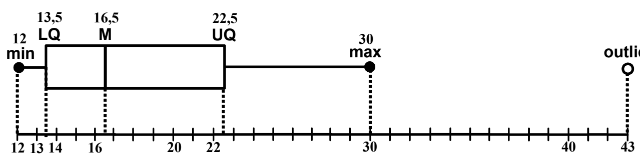


2(a) $Q_1 = 13,5$ $M = Q_2 = 16,5$

$Q_3 = 22,5$ $IQR = 9$

Interval for non-outliers $= [0 ; 36]$ Outlier is Manchester City

2(b)

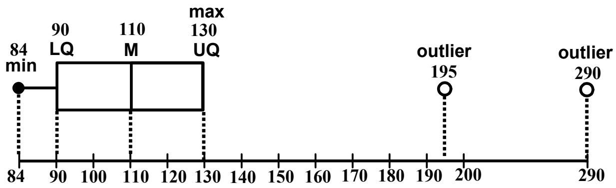


3(a) $Q_1 = 90$ $M = Q_2 = 110$ $Q_3 = 130$ $IQR = 40$

Interval for non-outliers $= [30\,;190]$

Celebrity 1 and 2 are outliers.

3(b)

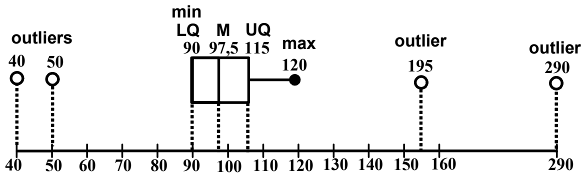


4(a) $Q_1 = 90$ $M = Q_2 = 97{,}5$ $Q_3 = 115$ $IQR = 25$

4(b) Interval for non-outliers $= [52{,}5\,;152{,}5]$

Outliers are 40 and 50 and 155

4(c)



EXERCISE 6

1(b) 27°C and 40°C **2(b)** Linear **2(c)** Approx 148 **3(c)** 4.0

3(d) Strong positive correlation **4(b)** Quadratic or exponential

4(c) 2 ; 11

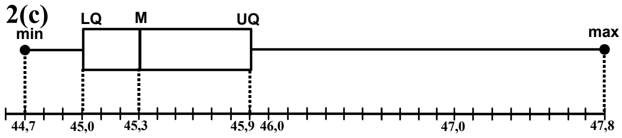
REVISION EXERCISE

1(a) median, max value **1(b)** No **1(c)** negatively skewed

1(d) 32 **1(e)** No

2(b) 45,5 **2(d)** positively skewed **2(e)** 0,7 **2(f)** 5 times **2(g)** 47,8

2(c)



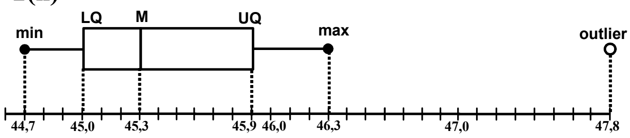
3(a) 500 **3(b)** 50 **3(c)** 450

3(d) 100 **3(e)** 1000 **3(f)** 250

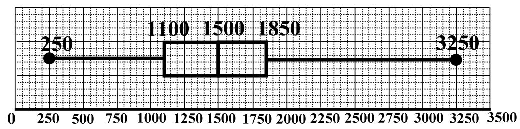
3(g) 40% **3(h)** 1500

3(k) 1540

2(h)



3(l)



3(m) Positively skewed

SOME CHALLENGES

1. Mean = 59 Median = 54 Standard deviation $= 17{,}4$ Range = 86
2. Standard deviation: 3 Variance: 9
3. $x = 9$
4. $x = 4$ $y = 8$
5. Therefore, one possible set of numbers could be:
   10 ; 20 ; 20 ; 32 ; 45 ; 49 ; 51 ; 53 ; 80
   Other possible values for *x*: $20 \le x \le 45$
   Other possible values for *y*: $45 \le x \le 51$
   where $x + y = 81$